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UCSB

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Investments: theory and practice

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References

Joint work with M. Musiela (BNP Paribas, London)

- "Investments and forward utilities"
  Preprint (2006)

- "Backward and forward dynamic utilities and their associated pricing systems: Case study of the binomial model"
  Indifference Pricing, PUP (2003, 2005)

- "Investment and valuation under backward and forward dynamic utilities in a stochastic factor model"
  to appear in Dilip Madan's Festschrift (2006)

  http://www.ma.utexas.edu/users/zariphop/

Topics

- Academia and financial industry
- Investment banking and utility theory
- Weaknesses and difficulties
- Alternative and more general approach
- Optimal asset allocations
- Portfolio dynamics and wealth distribution
- Practical applications

Derivatives and investments

Investment banking and martingale theory

- Ideal relationship
- Mathematical logic of the derivative business perfectly in line with the theory
- Pricing by replication comes down to the calculation of an expectation with respect to a martingale measure
- However, the modern investment banking is not about hedging (the essence of pricing by replication)
- Indeed, it is much more about return on capital - the business of hedging offers the lowest return

Investment banking and utility theory

- Dysfunctional relationship
- Mathematical utility theory formulated in a very abstract way and focused on solving problems of limited practical importance
- Economic utility theory formulated and developed in the context which is not directly focused on applications in investment banking
- When reformulated in the investment context, it faces the difficulty to explain the intuitive meaning of utility
- Only very sporadic examples where utility was used in a pricing context
- Limited use in the asset allocation context

Deterministic environment

$$u(x,t) \quad : \quad x \text{ **"wealth"** } \quad \text{and} \quad t \text{ **"time"**}$$

- Monotonicty $\quad u_x(x,t) > 0$

- Risk aversion $\quad u_{xx}(x,t) < 0$

- Impatience $\quad u_t(x,t) < 0$

Fisher (1913, 1918), Koopmans (1951),

Koopmans-Diamond-Williamson (1964) ...

Traditional framework

A deterministic utility datum $u(x)$ is assigned at the end of a fixed investment horizon

$$U(x,T) = u(x)$$

Backwards in time generation of optimal utility volume

$$V(x,t) = \sup_{\pi} E_{\mathbb{P}}(u(X_T^{\pi},T)|\mathcal{F}_t; X_t^{\pi} = x)$$

$$V(x,t) = \sup_{\pi} E_{\mathbb{P}}(V(X_s^{\pi},s)|\mathcal{F}_t; X_t^{\pi} = x) \qquad \text{(DPP)}$$

$$V(x,t) = E_{\mathbb{P}}(V(X_s^{\pi^*},s)|\mathcal{F}_t; X_t^{\pi^*} = x)$$

$$\Downarrow$$

$$U(x,t) \equiv V(x,t) \qquad 0 \le t < T$$

The dynamic utility coincides with the traditional value function

Weakness and difficulties

- No clear idea how to specify the utility function
- Explicit solutions to the optimal investment problems can be derived only under very restrictive model and utility assumptions - dependence on the Markovian assumption and HJB equations
- The general non Markovian models concentrate on the mathematical questions of existence of optimal allocations and on the dual representation of utility
- The utility at time 0, i.e., $U(x,0)$, may be very complicated and quite unintuitive
- Only very specific cases of such utilities, like exponential, can be analyzed in a model independent way
- No easy way to develop practical intuition for the asset allocation

An alternative and more general approach

Optimal performance criterion

$U(x,t)$ is an $\mathcal{F}_t$-adapted process

- As a function of x, U is increasing and concave
- For each self-financing strategy, represented by π, the associated (discounted) wealth X_t satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi},t) \mid \mathcal{F}_s) \leq U(X_s^{\pi},s) \qquad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by π^*, for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi^*},t) \mid \mathcal{F}_s) = U(X_s^{\pi^*},s) \qquad 0 \leq s \leq t$$

Optimal performance criterion

A deterministic target datum $u(x)$ is assigned at the beginning of the trading horizon, $t = 0$

$$U(x,0) = u(x)$$

Forward in time generation of optimal volume

$$U(X_s^{\pi^*}, s) = E_{\mathbb{P}}(U(X_t^{\pi^*}, t)|\mathcal{F}_s) \qquad 0 \leq s \leq t$$

- Optimal target/performance can be defined for all trading horizons
- Wealth and portfolios take a very intuitive form
- Difficulties due to the "inverse in time" nature of the problem

Dynamic measurement

time t_1, information $\mathcal{F}_{t_1}$

asset returns

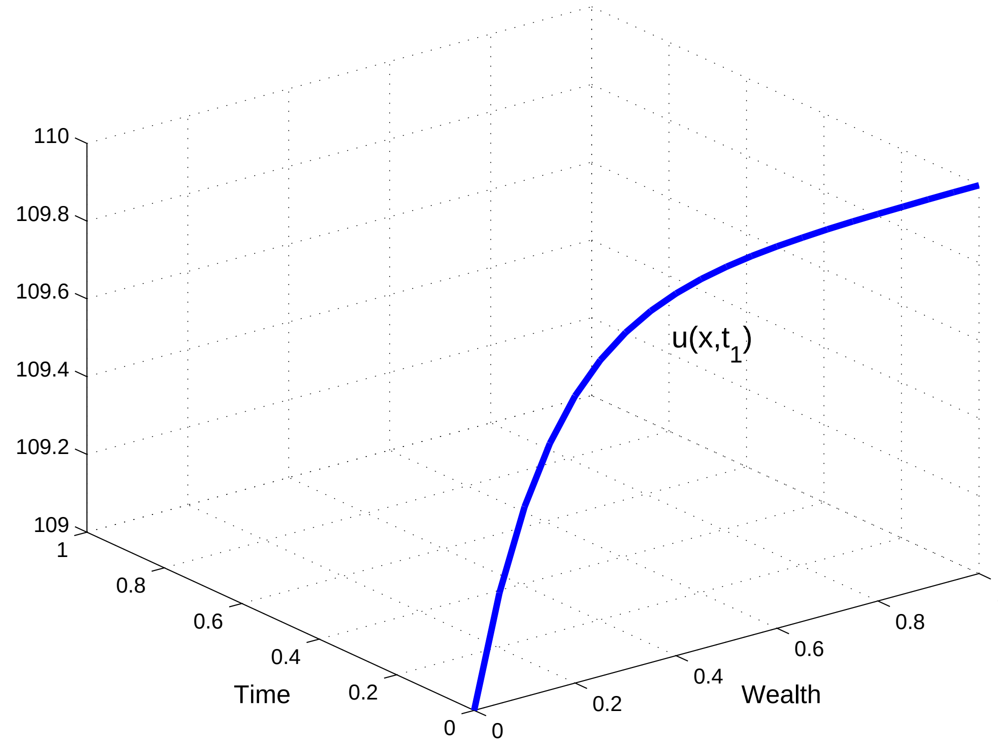
constraints

market view
away from equilibrium

benchmark numeraire

calendar time subordination

$\longrightarrow$



$$MI(t_1) \dashrightarrow + \dashleftarrow u(x,t_1)$$

$$\Downarrow$$

$$\boxed{U(x,t_1;MI) \in \mathcal{F}_{t_1} \qquad \pi(x,t_1;MI) \in \mathcal{F}_{t_1}}$$

Dynamic measurement

time t_2, information $\mathcal{F}_{t_2}$

asset returns

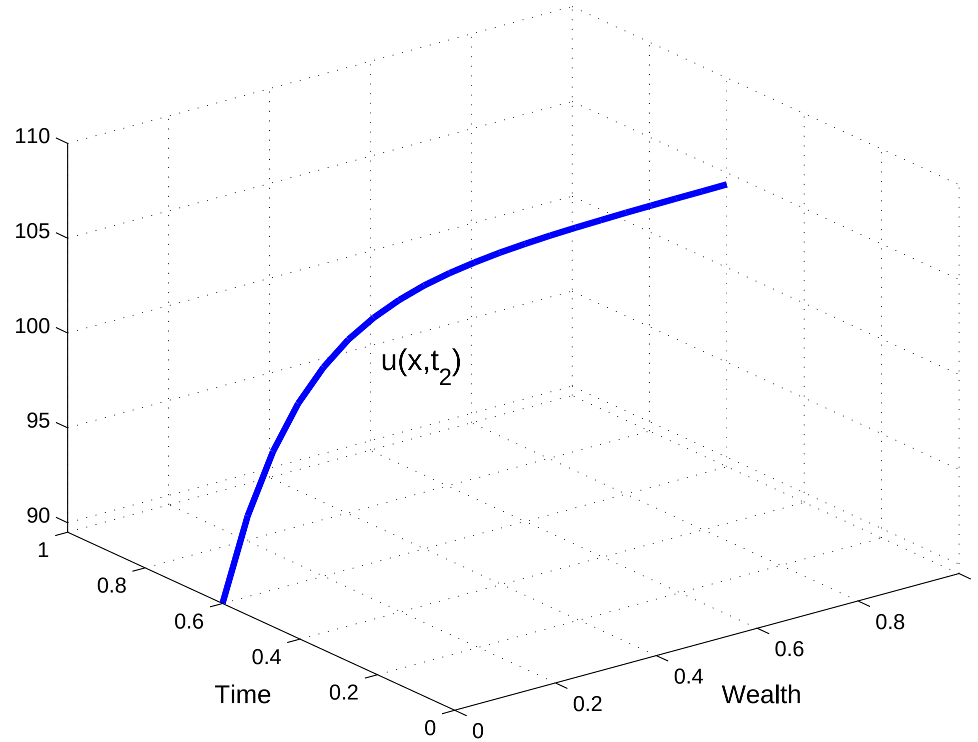
constraints

market view
away from equilibrium

benchmark   numeraire

calendar time   subordination

$\longrightarrow$



$$MI(t_2) \quad \dashrightarrow \quad + \quad \dashleftarrow \quad u(x, t_2)$$

$$\Downarrow$$

$$\boxed{U(x, t_2; MI) \in \mathcal{F}_{t_2} \qquad \pi(x, t_2; MI) \in \mathcal{F}_{t_2}}$$

Dynamic measurement

time t_3, information $\mathcal{F}_{t_3}$

asset returns

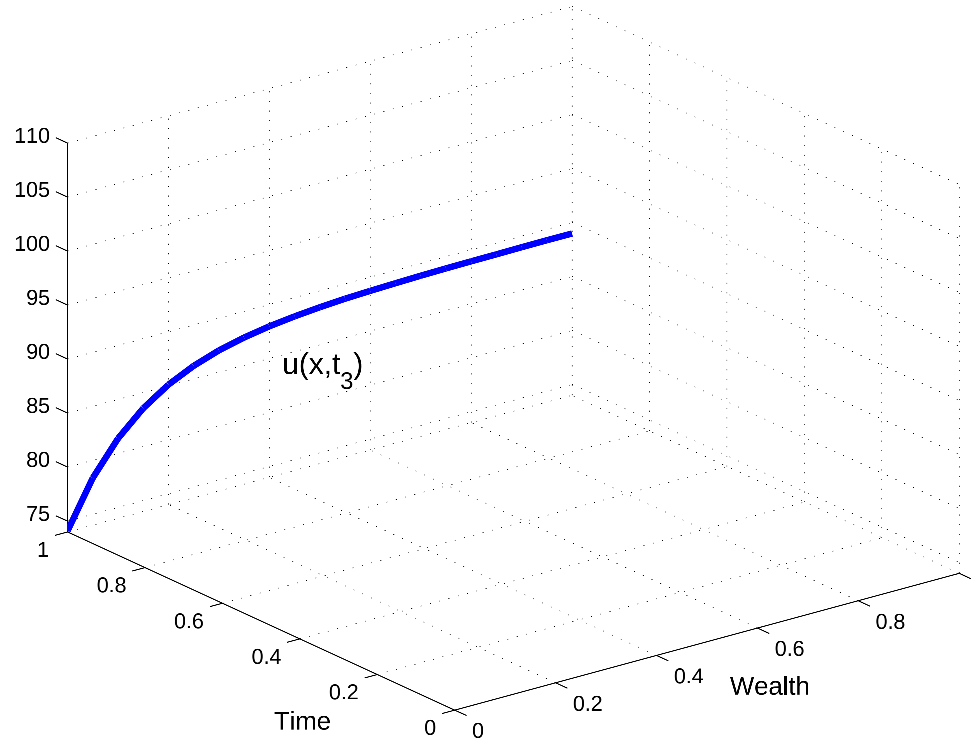
constraints

market view
away from equilibrium

benchmark numeraire

calendar time subordination

$\longrightarrow$



$$MI(t_3) \dashrightarrow + \dashleftarrow u(x, t_3)$$

$$\Downarrow$$

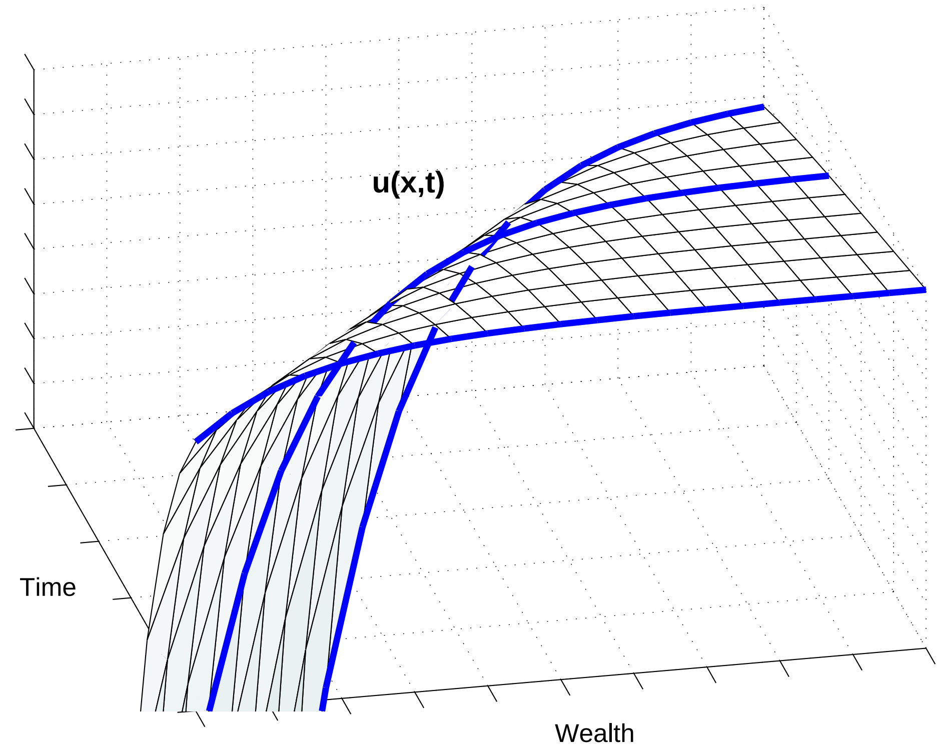
$$\boxed{U(x, t_3; MI) \in \mathcal{F}_{t_3} \qquad \pi(x, t_3; MI) \in \mathcal{F}_{t_3}}$$

Dynamic measurement

time t, information $\mathcal{F}_t$

asset returns
additional
market input

$\longrightarrow$



$$MI(t) \dashrightarrow + \dashleftarrow u(x,t)$$

$$\Downarrow$$

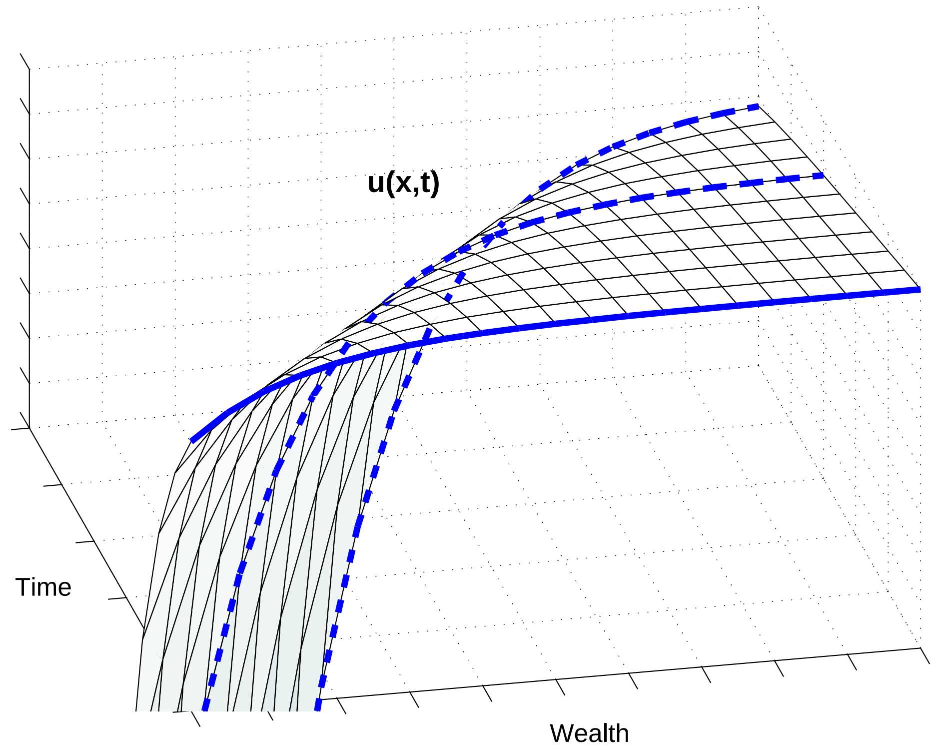
$$U(X_t^*, t) \in \mathcal{F}_t \qquad \pi^*(X_t^*, t) \in \mathcal{F}_t$$

Dynamic measurement

time t_1, information $\mathcal{F}_{t_1}$

asset returns
additional
market input

$\longrightarrow$



$$MI(t_1) \dashrightarrow + \dashleftarrow u(x, t_1)$$

$$\Downarrow$$

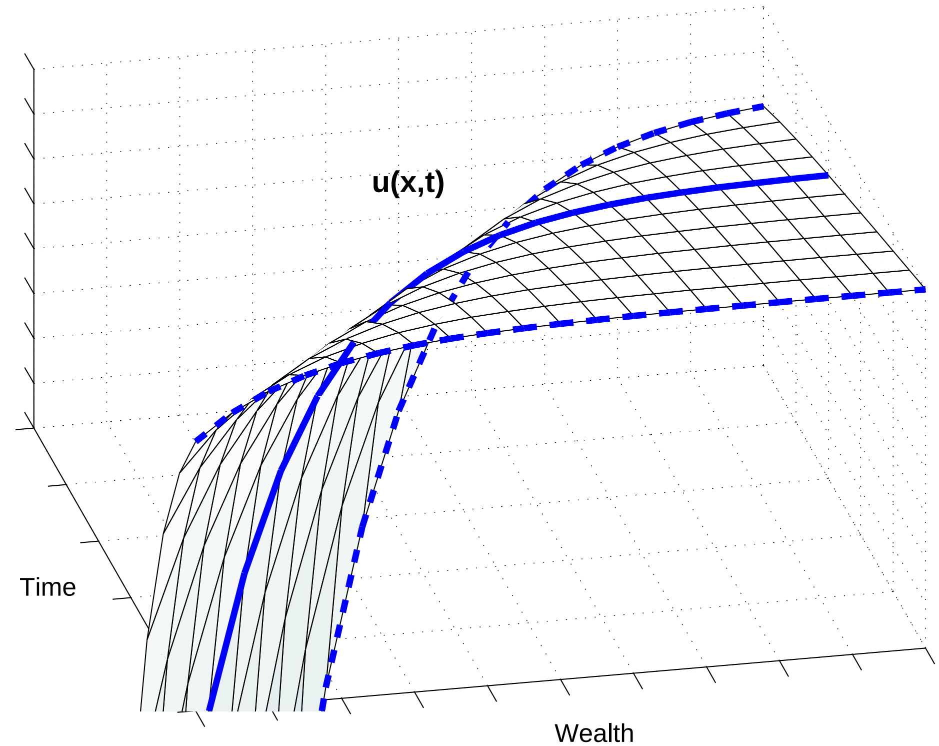
$$U(X^*_{t_1}, t_1) \in \mathcal{F}_{t_1} \qquad \pi^*(X^*_{t_1}, t_1) \in \mathcal{F}_{t_1}$$

Dynamic measurement

time t_2, information $\mathcal{F}_{t_2}$

asset returns
additional
market input

$\longrightarrow$



$$MI(t_2) \dashrightarrow + \dashleftarrow u(x,t_2)$$

$$\Downarrow$$

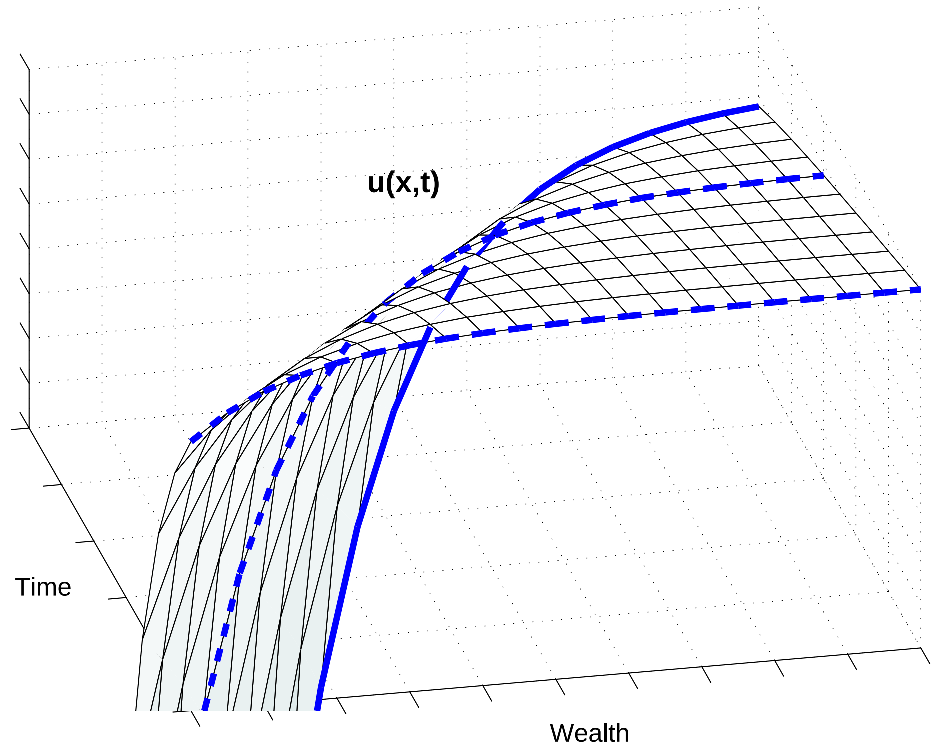
$$U(X^*_{t_2}, t_2) \in \mathcal{F}_{t_2} \qquad \pi^*(X^*_{t_2}, t_2) \in \mathcal{F}_{t_2}$$

Dynamic measurement

time t_3, information $\mathcal{F}_{t_3}$

asset returns
additional
market input

$\longrightarrow$



$$MI(t_3) \quad \dashrightarrow \quad + \quad \dashleftarrow \quad u(x, t_3)$$

$$\Downarrow$$

$$U(X^*_{t_3}, t_3) \in \mathcal{F}_{t_3} \qquad \pi^*(X^*_{t_3}, t_3) \in \mathcal{F}_{t_3}$$

Important ingredients of "forward" performance measurement

- Time evolution concurrent with the one of the investment/opportunities universe

- Function $U(x, 0)$ represents performance of today and not for, say, ten years ahead

- Consistency with up to date information

- Incorporation of available opportunities, constraints and views

- Construction of a rich class of solutions that yield explicit optimal allocations under minimal model assumptions

Construction of a rich class of forward target/performance processes

Creating the martingale that yields the optimal performance

Minimal model assumptions

Stochastic optimization problem “inverse” in time

Key idea

Stochastic input — **Market**

Variational input — **Individual**

Maximal performance — Optimal allocation

Variational input

Variational input

A model independent differential constraint on
impatience, risk aversion and monotonicity

- Initial datum

$$u_0(x) = u(x, 0)$$

- Fully non-linear pde

$$\begin{cases} u_t \ u_{xx} = \frac{1}{2} u_x^2 \\ u(x, 0) = u_0(x) \end{cases}$$

Intuition for the above pde comes from a variety of examples
in distinct model settings

Transport equation

The "utility" equation can be alternatively viewed as a transport equation with slope of its characteristics equal to (half of) the risk tolerance

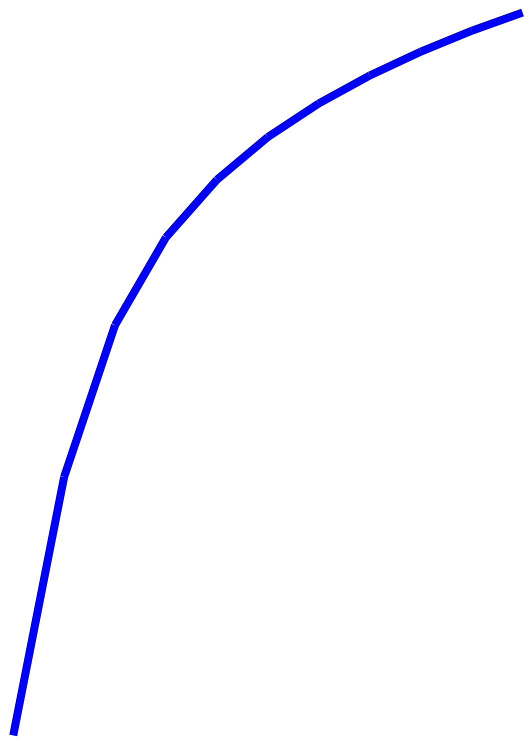
$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

$$\begin{cases} u_t + \frac{1}{2} r(x,t) u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Characteristic curves: $$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t),t)$$

Construction of surface $u(x,t)$ using characteristics

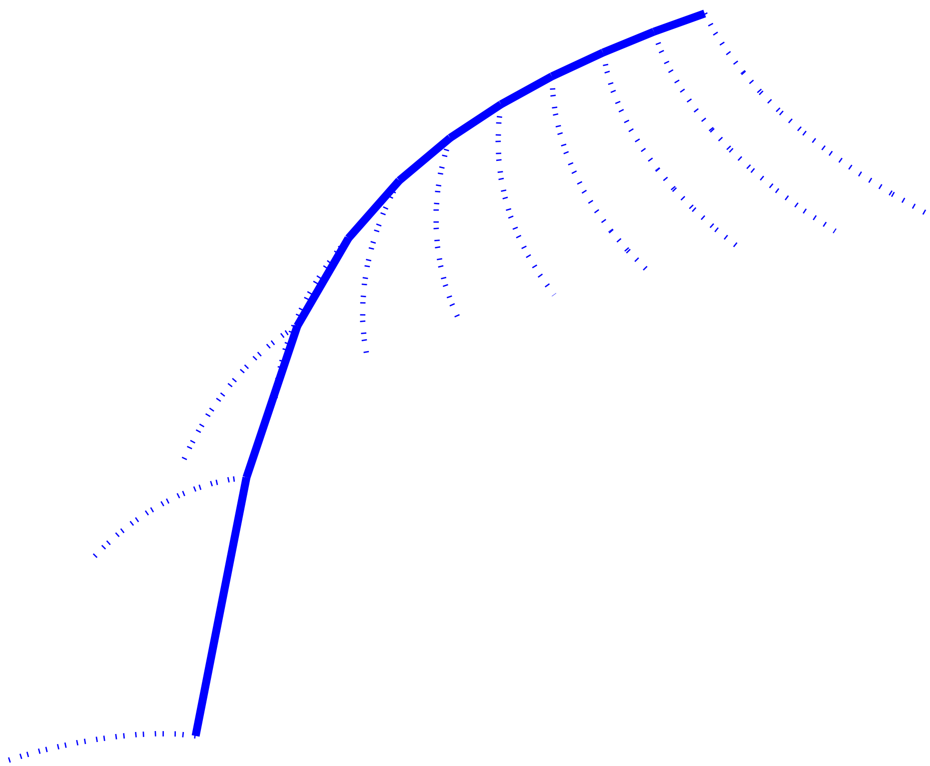
$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t)$$



Initial datum $u_0(x)$

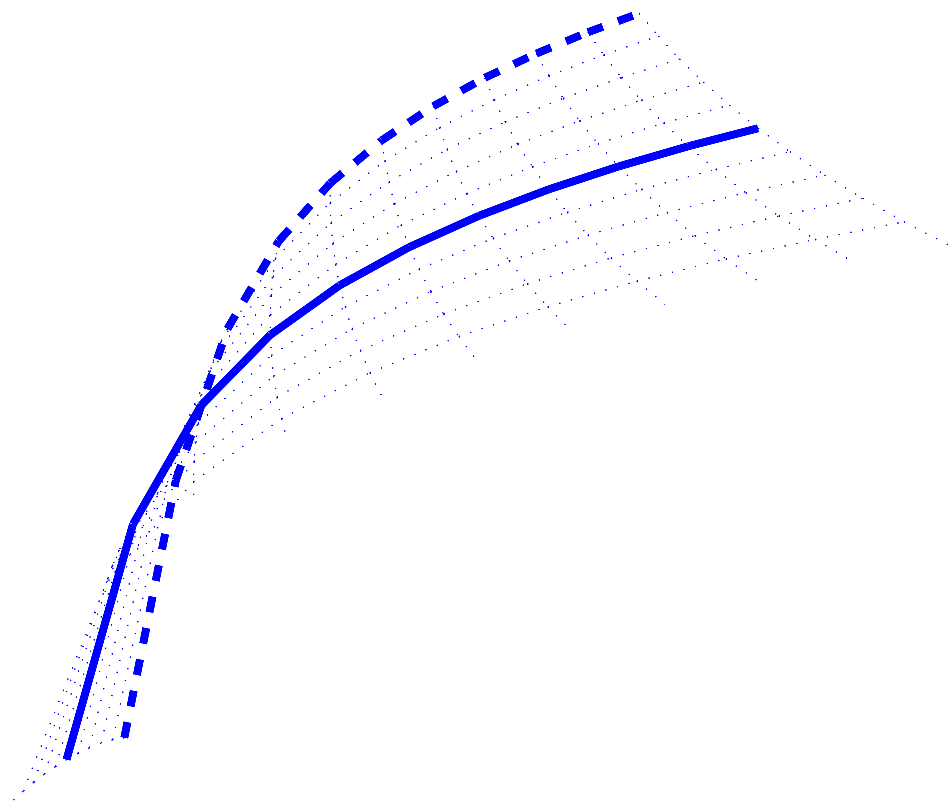
Construction of characteristics

$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t)$$

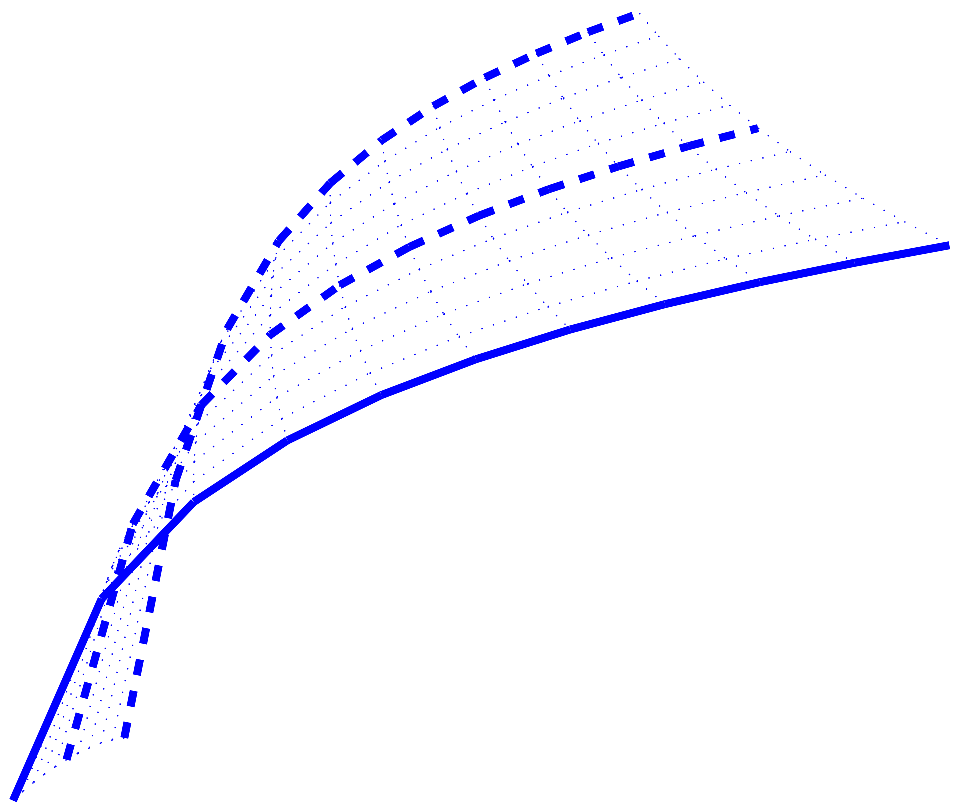


Initial datum $u_0(x)$
Characteristic curves

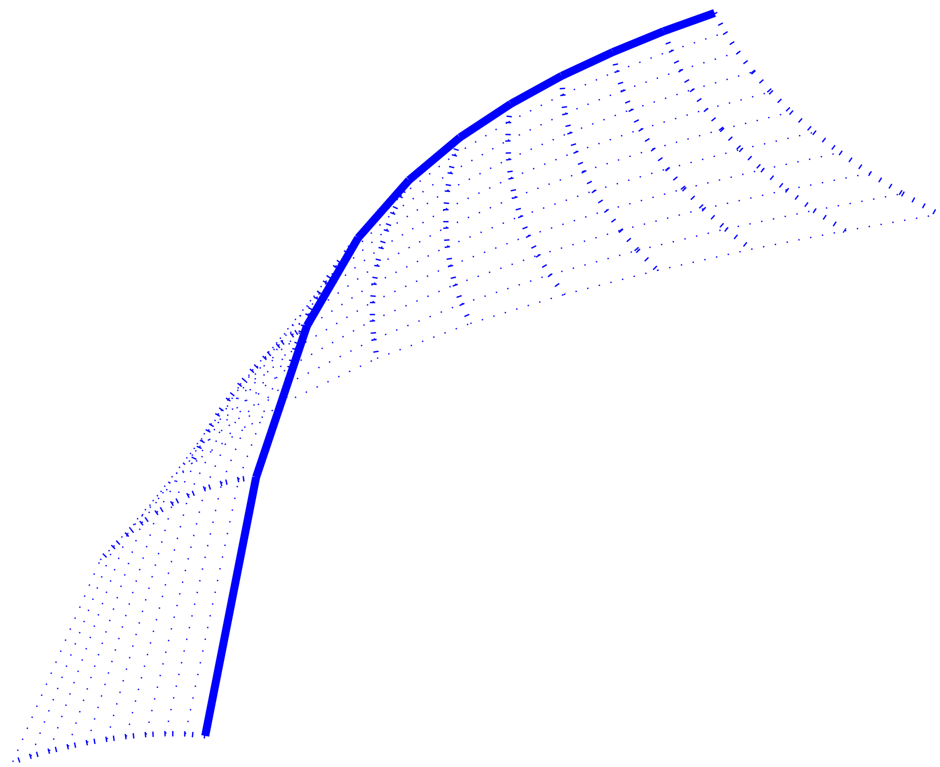
Propagation of initial datum along characteristics



Propagation of initial datum along characteristics



Variational input $u(x,t)$



The risk tolerance pde

- Recall the equation

$$\begin{cases} u_t \ u_{xx} = \dfrac{1}{2} u_x^2 \\ \\ u(x,0) = u_0(x) \end{cases}$$

- The local risk tolerance $\boldsymbol{r(x,t) = -u_x(x,t)/u_{xx}(x,t)}$ solves the autonomous equation of "fast diffusion type"

$$\begin{cases} r_t + \frac{1}{2} r^2 r_{xx} = 0 \\ \\ r(x,0) = r_0(x) \end{cases} \qquad \text{(FDE)}$$

The risk aversion pde

$$\gamma(x,t) = \frac{1}{r(x,t)}$$

Porous medium equation

$$\begin{cases} \gamma_t = \left(\dfrac{1}{\gamma}\right)_{xx} \\ \gamma(x,0) = \dfrac{1}{r_0(x)} \end{cases} \qquad \text{(PME)}$$

Risk tolerance/risk aversion pdes

$$n = 2$$

$$\gamma_t + \nabla(\gamma^{-2}\nabla\gamma) = 0$$

$$\Updownarrow \, r = \gamma^{-1}$$

$$r_t + \frac{1}{2}r^2 r_{xx} = 0$$

Difficulties

- Very limited results for $n > 1$
- Solutions blow up in finite time
- Inverse in time problem

Stochastic market input

Market input

- Investment universe of one riskless and k risky securities
- General Ito type dynamics for the risky securities
- Standard d-dimensional Brownian motion driving the dynamics of the traded assets
- Traded assets dynamics

$$\begin{cases} dS_t^i = S_t^i(\mu_t^i dt + \sigma_t^i \cdot dW_t), \quad i = 1, \cdots, k \\ dB_t = r_t B_t dt \end{cases}$$

Market input

- Assume existence of the market price for risk process which satisfies

$$\mu_t - r_t \mathbf{1} = \sigma_t^T \lambda_t$$

- Benchmark process

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t), \quad Y_0 = 1, \quad \sigma_t \sigma_t^+ \delta_t = \delta t$$

- Views (constraints) process

$$dZ_t = Z_t \phi_t \cdot dW_t, \quad Z_0 = 1$$

- Subordination process

$$dA_t = |\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t|^2 dt, \quad A_0 = 0$$

Optimal performance and asset allocation

Optimal performance

Stochastic market input

$\lambda_t, \ \sigma_t$

$\downarrow$

benchmark, views

subordination

$(Y_t, \ Z_t, \ A_t)$

Variational input

$$x, r_0(x) = -\frac{u_0'(x)}{u_0''(x)}$$

$\downarrow$

$r_t + \frac{1}{2} r^2 r_{xx} = 0$ (FDE)

$u_t + \frac{1}{2} r u_x = 0$ (TE)

$u(x, t)$

$$U(x,t) = u\left(\frac{x}{A_t}, Y_t\right) Z_t$$

Minimal model assumptions!

The structure of optimal portfolios

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t \, dt + dW_t)$$

Stochastic input
Market

$(Y_t,\, Z_t,\, A_t)$
$\lambda_t,\, \sigma_t,\, \delta_t,\, \phi_t$

Variational input
Individual

wealth x
risk tolerance $r(x,t)$

$\frac{1}{Y_t}\pi_t^*$ is a **linear** combination

of (benchmarked) optimal wealth

and subordinated (benchmarked) risk tolerance

Optimal Portfolio

- The optimal portfolio is given by

$$\frac{1}{Y_t}\pi_t^* = \sigma_t^+ \left[\left(\frac{X_t^*}{Y_t} - R_t^* \right) \delta_t + R_t^* (\lambda_t + \phi_t) \right]$$

$$R_t^* = r\left(\frac{X_t^*}{Y_t}, A_t \right)$$

$$r_t + \frac{1}{2} r^2 r_{xx} = 0 \qquad r(x, 0) = r_0(x)$$

- The optimal wealth, the associated risk tolerance and the optimal allocations are benchmarked
- The optimal portfolio incorporates the investor views or constraints on top of the market equilibrium
- The optimal portfolio depends on the investor risk tolerance at time 0

Wealth and risk tolerance become the important state variables

Wealth and risk tolerance dynamics

- The dynamics of the (benchmarked) optimal wealth and risk tolerance are given by

$$\begin{cases} d\left(\frac{X_t^*}{Y_t}\right) = R_t^*(\sigma_t\sigma_t^+(\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t)dt + dW_t) \\ dR_t^* = r_x\left(\frac{X_t^*}{Y_t}, A_t\right) d\left(\frac{X_t^*}{Y_t}\right) \end{cases}$$

- Observe that zero risk tolerance translates to following the benchmark and generating pure beta exposure.

Canonical dynamics

- The previous system of equations reduces to

$$\begin{cases} dx_1(t) = x_2(t)dw(t), \quad x_1(0) = z \quad z = \frac{x}{y} \\ dx_2(t) = r_x(x_1(t), t)x_2(t)dw(t), \quad x_2(0) = r_0(z) \end{cases}$$

- It turns out that it can be solved analytically

One can easily revert to the original coordinates to obtain explicit expressions for $\frac{X_t^*}{Y_t}$, R_t^* and π_t^*

Investment scenaria

No benchmark and no views

$$\phi_t = 0, \quad \delta_t = 0$$

- The optimal allocations, given below, are expressed in the discounted with the riskless asset amounts

$$\pi_t^* = R_t^* \sigma_t^+ \lambda_t, \quad R_t^* = r(X_t^*, A_t)$$

$$dA_t = |\sigma_t \sigma^* \lambda_t|^2 dt$$

$$r_t + \frac{1}{2} r^2 r_{xx} = 0, \quad r(x, 0) = r_0(x)$$

No benchmark and no views

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$$dA_t = |\sigma_t \sigma^* \lambda_t|^2 dt$$

$$r_t + \frac{1}{2} r^2 r_{xx} = 0, \quad r(x,0) = r_0(x)$$

- They depend on the market price of risk, asset volatilities and the investor's risk tolerance at time 0

- Observe the no direct dependence on the "utility" function and the link between the distribution of the optimal discounted wealth in the future and the implicit to it current risk tolerance of the investor

No benchmark and hedging constraints

The derivatives business can be seen from the investment perspective as an activity for which it is optimal to hold a portfolio which earns riskless rate

- By formulating views against market equilibrium one takes a risk neutral position and allocates zero wealth to the risky investment, indeed,

$$\delta_t = 0, \qquad \phi_t = -\lambda_t \quad \Longrightarrow \quad \pi_t^* = 0$$

- Other constraints can also be incorporated by the appropriate specification of the benchmark and of the vector for market views

No riskless allocation

- Take a vector such that

$$\mathbf{1} \cdot \sigma_t^+ \nu_t \neq 0$$

- Choose

$$\phi_t = \frac{\mathbf{1} - \mathbf{1} \cdot \sigma_t^+ \lambda_t}{\mathbf{1} \cdot \sigma_t^+ \nu_t} \nu_t, \quad \delta_t = \sigma_t \sigma_t^+ (\lambda_t + \phi_t)$$

- The optimal allocation is given by

$$\pi_t^* = X_t^* \sigma_t^+ (\lambda_t + \phi_t)$$

- It puts zero wealth into the riskless investment, indeed,

$$\mathbf{1} \cdot \pi_t^* = X_t^* \mathbf{1} \cdot \sigma_t^+ \left(\lambda_t + \frac{\mathbf{1} - \mathbf{1} \cdot \sigma_t^+ \lambda_t}{\mathbf{1} \cdot \sigma_t^+ \nu_t} \nu_t \right) = X_t^*$$

Summary of how the "forward" performance approach works

Steps to follow

- Specify the investment universe and its equilibrium dynamics
- Determine the current risk tolerance of an investor relatively to that universe (could try to imply it from the specification of future wealth distribution)
- Specify a benchmark and views or constraints
- Solve the FDE to recover the function $r(x,t)$
- Determine the variational input $u(x,t)$, namely, calculate

$$u(x,t) = \int_{\hat{x}}^{x} \exp\left(-\int_0^y \frac{dz}{r(z,t)} + K_1(t)\right)dy + K_2(t)$$

Steps to follow (continued)

- Specify the forward optimal target process by combining the variational input with the choice of a benchmark, views or constraints

$$U(x,t) = u\left(\frac{x}{Y_t}, A_t\right) Z_t$$

- Specify the risk tolerance, the optimal wealth and the optimal portfolio

$$R_t = r\left(\frac{X_t^*}{Y_t}, A_t\right)$$

$$\pi_t^* = MI_1(t)X_t^* + MI_2(t)R_t^*$$

- Analyze the outcome and potentially recalibrate

From theory to investment practice

Steps to follow (work in progress)

- Investor specifies a desired future wealth distribution
- Inference of the investor's future risk tolerance
- Construction of local risk tolerance via FDE
- Specification of benchmark and/or views, constraints
- Construction of optimal target process
- Construction of optimal portfolio and its dynamics
- Manager determines the required capital and specifies the risk budget