PDE Approach to Credit Derivatives

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THE MODEL

Default Time

- The default time τ is a non-negative random variable on $(\Omega, \mathcal{G}, \mathbb{Q})$.
- Note that $\mathbb{Q}$ is the statistical probability measure.
- The filtration generated by the default process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ is denoted by $\mathbb{H}$.
- We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for every $t \in \mathbb{R}_+$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a reference filtration.
- We define the processes F_t and G_t as

$$F_t = \mathbb{Q}\{\tau \leq t \,|\, \mathcal{F}_t\}$$

and

$$G_t = 1 - F_t = \mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}.$$

Hazard Process

- The process Γ, given as

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t$$

  is the $\mathbb{F}$-hazard process under the statistical probability $\mathbb{Q}$.
- We shall assume that the $\mathbb{F}$-hazard process is absolutely continuous: $\Gamma_t = \int_0^t \gamma_u \, du$.
- Hence, the compensated default process

$$M_t = H_t - \int_0^{t \wedge \tau} \gamma_u \, du = H_t - \int_0^t \xi_u \, du,$$

  is a $\mathbb{G}$-martingale under $\mathbb{Q}$, where we denote $\xi_t = \gamma_t \mathbb{1}_{\{t < \tau\}}$.

Hypothesis (H)

Hypothesis (H). We assume throughout that any $\mathbb{F}$-martingale under $\mathbb{Q}$ is also a $\mathbb{G}$-martingale under $\mathbb{Q}$.

- Hypothesis (H) is satisfied if a random time τ is defined through the canonical construction.
- If the representation theorem holds for the filtration $\mathbb{F}$ and a finite family $Z^i, i \leq n$, of $\mathbb{F}$-martingales then, under Hypothesis (H), it holds also for the filtration $\mathbb{G}$ and with respect to the $\mathbb{G}$-martingales $Z^i, i \leq n$ and M.

Remark. Hypothesis (H) is not invariant with respect to an equivalent change of a probability measure, in general.

Dynamics of Traded Assets

- Let Y^1, Y^2, Y^3 be semimartingales on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$. We interpret Y^i_t as the cash price at time t of the ith traded asset in the market model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$, where Φ stands for the class of all self-financing trading strategies.
- We postulate that the process Y^i is governed by the SDE

$$dY^i_t = Y^i_{t-}\big(\mu_i\, dt + \sigma_i\, dW_t + \kappa_i\, dM_t\big), \quad i = 1, 2, 3,$$

  with $Y^i_0 > 0$.
- Here W is a one-dimensional Brownian motion and the M is the compensated martingale of the default process H.

Assumptions

- We assume that that $\kappa_i \geq -1$ and $\kappa_1 > -1$ so that $Y^1_t > 0$ for every $t \in \mathbb{R}_+$. This assumptions allows us to take the first asset as a numeraire.
- Note that the constant coefficient $\kappa_1 > -1$ corresponds to a fractional recovery of market value for the first asset.
- In general, we do not assume that a risk-free security exists. Hence we do not refer to the theory involving the risk-neutral probability associated with the choice of a savings account as a numeraire.

Change of a Numeraire

- An equivalent martingale measure $\widetilde{\mathbb{Q}}$ is characterized by the property that the relative prices $Y^i(Y^1)^{-1}$, $i = 1, 2, 3$, are $\widetilde{\mathbb{Q}}$-martingales.
- We will derive the dynamics for the process $Y^{i,1} = Y^i(Y^1)^{-1}$ for $i = 2, 3$.
- From Itô's formula, we first obtain

$$d\left(\frac{1}{Y_t^1}\right) = \frac{1}{Y_{t-}^1}\left(-\mu_1 + \sigma_1^2 + \xi_t\left(\frac{1}{1+\kappa_1} - 1 + \kappa_1\right)\right) dt - \frac{1}{Y_{t-}^1}\left(\sigma_1 dW_t + \frac{\kappa_1}{1+\kappa_1}\, dM_t\right).$$

Dynamics of Relative Prices

Consequently, the Itô's integration by parts formula yields the following dynamics for the processes $Y^{i,1}$

$$
\begin{aligned}
dY_t^{i,1} &= Y_{t-}^{i,1}\Big\{ \Big(\mu_i - \mu_1 - \sigma_1(\sigma_i - \sigma_1) - \xi_t(\kappa_i - \kappa_1)\frac{\kappa_1}{1+\kappa_1}\Big)\, dt \\
&+ (\sigma_i - \sigma_1)\, dW_t + \frac{\kappa_i - \kappa_1}{1+\kappa_1}\, dM_t \Big\}.
\end{aligned}
$$

Equivalent Martingale Measure

- By assumption, $\widetilde{\mathbb{Q}}$ is equivalent to the statistical probability $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ and such that $Y^{i,1}$, $i = 2, 3$ are $\widetilde{\mathbb{Q}}$-martingales.
- Kusuoka (1999) showed that any probability equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ is defined by means of its Radon-Nikodým density process η satisfying the SDE
$$d\eta_t = \eta_{t-}\big(\theta_t\, dW_t + \zeta_t\, dM_t\big), \quad \eta_0 = 1,$$
where θ and ζ are $\mathbb{G}$-predictable processes satisfying mild technical conditions (in particular, $\zeta_t > -1$ for $t \in [0, T]$).
- Since M is stopped at τ, we may and do assume that ζ is stopped at τ.

Radon-Nikodým Density

We define $\widetilde{\mathbb{Q}}$ by setting

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \eta_T = \mathcal{E}_T(\theta W)\mathcal{E}_T(\zeta M), \quad \mathbb{Q}\text{-a.s.}$$

Then the processes $\widehat{W}$ and $\widehat{M}$ given by, for $t \in [0, T]$,

$$\begin{aligned} \widehat{W}_t &= W_t - \int_0^t \theta_u \, du, \\ \widehat{M}_t &= M_t - \int_0^t \xi_u \zeta_u \, du = H_t - \int_0^t \xi_u (1 + \zeta_u) \, du = H_t - \int_0^t \widehat{\xi}_u \, du, \end{aligned}$$

where $\widehat{\xi}_u = \xi_u(1 + \zeta_u)$, are $\mathbb{G}$-martingales under $\widetilde{\mathbb{Q}}$.

Martingale Condition

Proposition

Processes $Y^{i,1}$, $i = 2, 3$ are $\widetilde{\mathbb{Q}}$-martingales if and only if drifts in their dynamics, when expressed in terms of $\widehat{W}$ and $\widehat{M}$, vanish.

Hence the following equalities hold for $i = 2, 3$ and every $t \in [0, T]$

$$Y^{i,1}_{t-}\left\{\mu_1 - \mu_i + (\sigma_1 - \sigma_i)(\theta_t - \sigma_1) + \xi_t(\kappa_1 - \kappa_i)\frac{\zeta_t - \kappa_1}{1 + \kappa_1}\right\} = 0.$$

Equivalently, we have for $i = 2, 3$, on the set $Y^{i,1}_{t-} \neq 0$,

$$\mu_1 - \mu_i + (\sigma_1 - \sigma_i)(\theta_t - \sigma_1) + \xi_t(\kappa_1 - \kappa_i)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} = 0.$$

CASE A: STRICTLY POSITIVE PRIMARY ASSETS

Case A: Strictly Positive Primary Assets

Case A: standing assumptions:

- We postulate that $\kappa_1 > -1$ so that $Y^1 > 0$.
- We assume, in addition, that $\kappa_i > -1$ for $i = 2, 3$, so that the price processes Y^2 and Y^3 are strictly positive as well.

Martingale condition:

- From the general theory of arbitrage pricing, it follows that the market model $\mathcal{M}$ is complete and arbitrage-free if there exists a unique solution (θ, ζ) such that the process $\zeta > -1$.
- Since $Y^{i,1} > 0$, we search for processes (θ, ζ) such that for $i = 2, 3$

$$\theta_t(\sigma_1 - \sigma_i) + \zeta_t \xi_t \frac{\kappa_1 - \kappa_i}{1 + \kappa_1} = \mu_i - \mu_1 + \sigma_1(\sigma_1 - \sigma_i) + \xi_t(\kappa_1 - \kappa_i) \frac{\kappa_1}{1 + \kappa_1}.$$

Martingale Condition

Since $\xi_t = \gamma \mathbb{1}_{\{t \leq \tau\}}$, we deal here with four linear equations.

- For $t \leq \tau$:

$$\begin{aligned}\theta_t(\sigma_1 - \sigma_2) + \zeta_t \gamma \frac{\kappa_1 - \kappa_2}{1 + \kappa_1} &= \mu_2 - \mu_1 + \sigma_1(\sigma_1 - \sigma_2) + \gamma \frac{(\kappa_1 - \kappa_2)\kappa_1}{1 + \kappa_1}, \\ \theta_t(\sigma_1 - \sigma_3) + \zeta_t \gamma \frac{\kappa_1 - \kappa_3}{1 + \kappa_1} &= \mu_3 - \mu_1 + \sigma_1(\sigma_1 - \sigma_3) + \gamma \frac{(\kappa_1 - \kappa_3)\kappa_1}{1 + \kappa_1}.\end{aligned}$$

- For $t > \tau$:

$$\begin{aligned}\theta_t(\sigma_1 - \sigma_2) &= \mu_2 - \mu_1 + \sigma_1(\sigma_1 - \sigma_2), \\ \theta_t(\sigma_1 - \sigma_3) &= \mu_3 - \mu_1 + \sigma_1(\sigma_1 - \sigma_3).\end{aligned}$$

- The first (the second, resp.) pair of equations is referred to as the pre-default (post-default, resp.) no-arbitrage condition.

Notation

To solve explicitly these equations, we find it convenient to write

$$a = \det A, \quad b = \det B, \quad c = \det C,$$

where A, B and C are the following matrices:

$$A = \begin{bmatrix} \sigma_1 - \sigma_2 & \kappa_1 - \kappa_2 \\ \sigma_1 - \sigma_3 & \kappa_1 - \kappa_3 \end{bmatrix}, \quad B = \begin{bmatrix} \sigma_1 - \sigma_2 & \mu_1 - \mu_2 \\ \sigma_1 - \sigma_3 & \mu_1 - \mu_3 \end{bmatrix},$$

$$C = \begin{bmatrix} \kappa_1 - \kappa_2 & \mu_1 - \mu_2 \\ \kappa_1 - \kappa_3 & \mu_1 - \mu_3 \end{bmatrix}.$$

Auxiliary Lemma

Lemma

The pair (θ, ζ) *satisfies the following equations*

$$\begin{aligned} \theta_t a &= \sigma_1 a + c, \\ \zeta_t \xi_t a &= \kappa_1 \xi_t a - (1 + \kappa_1) b. \end{aligned}$$

In order to ensure the validity of the second equation after the default time τ (i.e., on the set $\{\xi_t = 0\}$), we need to impose an additional condition, $b = 0$, or more explicitly,

$$(\sigma_1 - \sigma_2)(\mu_1 - \mu_3) - (\sigma_1 - \sigma_3)(\mu_1 - \mu_2) = 0.$$

If this holds, then we obtain the following equations

$$\begin{aligned} \theta_t a &= \sigma_1 a + c, \\ \zeta_t \xi_t a &= \kappa_1 \xi_t a. \end{aligned}$$

Existence of a Martingale Measure

Proposition

(i) *If* $a \neq 0$ *and* $b = 0$ *then the unique martingale measure* $\widetilde{\mathbb{Q}}$ *has the Radon-Nikodým density of the form*

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \mathcal{E}_T(\theta W)\mathcal{E}_T(\zeta M),$$

where the constants θ *and* ζ *are given by*

$$\theta = \sigma_1 + \frac{c}{a}, \quad \zeta = \kappa_1 > -1,$$

and where we write, for $t \in [0, T]$,

$$\mathcal{E}_t(\theta W) = \exp\left(\theta W_t - \frac{1}{2}\theta^2 t\right)$$

$$\mathcal{E}_t(\zeta M) = \left(1 + \mathbb{1}_{\{\tau \le t\}}\zeta\right)\exp\left(-\zeta\gamma(t \wedge \tau)\right).$$

Existence of a Martingale Measure

Proposition

(ii) *If $a \neq 0$ and $b = 0$ then the model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is arbitrage-free and complete. Moreover, the process (Y^1, Y^2, Y^3, H) has the Markov property under $\widetilde{\mathbb{Q}}$.*

(iii) *If $a = 0$ and $b = 0$ then a solution (θ, ζ) exists provided that $c = 0$ and the uniqueness of a martingale measure $\widetilde{\mathbb{Q}}$ fails to hold. In this case, the model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is arbitrage-free, but it is not complete.*

(iv) *If $b \neq 0$ then a martingale measure fails to exist and consequently the model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is not arbitrage-free.*

Example A: Extension of the Black-Scholes Model

- Assume that the asset Y^1 is risk-free, the asset $Y^2 \neq Y^1$ is default-free, and Y^3 is a defaultable asset with non-zero recovery, so that

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\left(\mu_2\,dt + \sigma_2\,dW_t\right), \\ dY_t^3 &= Y_{t-}^3\left(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t\right). \end{aligned}$$

- We thus have $\sigma_1 = \kappa_1 = 0$, $\mu_1 = r$, $\sigma_2 \neq 0$, $\kappa_2 = 0$, and $\kappa_3 \neq 0$, $\kappa_3 > -1$.
- Therefore,

$$a = \sigma_2\kappa_3 \neq 0, \quad c = \kappa_3(r - \mu_2),$$

and the equality $b = 0$ holds if and only if

$$\sigma_2(r - \mu_3) = \sigma_3(r - \mu_2).$$

Example A (Continued)

- It is easy to check that

$$\theta = \frac{r - \mu_2}{\sigma_2}, \quad \zeta = 0,$$

  and thus under the martingale measure $\widetilde{\mathbb{Q}}$ we have (irrespective of whether $\sigma_3 > 0$ or $\sigma_3 = 0$)

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(r\,dt + \sigma_2\,d\widehat{W}_t\big), \\ dY_t^3 &= Y_{t-}^3\big(r\,dt + \sigma_3\,d\widehat{W}_t + \kappa_3\,dM_t\big). \end{aligned}$$

- Since $\zeta = 0$ the risk-neutral default intensity $\widehat{\gamma}$ coincides here with the statistical default intensity γ. This implies the equality $\widehat{M} = M$.

CASE B: DEFAULTABLE ASSET WITH ZERO RECOVERY

Case B: Defaultable Asset with Zero Recovery

Case B: standing assumptions:

- We postulate that $\kappa_i > -1$ for $i = 1, 2$ and $\kappa_3 = -1$.
- This implies that the price of a defaultable asset Y^3 vanishes after τ, and thus the findings of the preceding section are no longer valid.

Martingale condition:

- Since Y^3 jumps to zero at τ, the first equality in the martingale condition

$$\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) + \xi_t(\kappa_2 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} = 0$$

  should still be satisfied for every $t \in [0, T]$.
- The second equality in the martingale condition

$$\mu_3 - \mu_1 + (\sigma_3 - \sigma_1)(\theta_t - \sigma_1) + \xi_t(\kappa_3 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} = 0$$

  is required to hold on the set $\{\tau > t\}$ only (i.e. when $\xi_t = \gamma$).

Martingale Condition

Lemma

Under the present assumptions, the unknown processes θ and ζ in the Radon-Nikodým density of $\widetilde{\mathbb{Q}}$ with respect to $\mathbb{Q}$ satisfy the following equations

$$\begin{aligned}
\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) &= 0, \quad \textit{for } t > \tau, \\
\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) + \gamma(\kappa_2 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} &= 0, \quad \textit{for } t \le \tau, \\
\mu_3 - \mu_1 + (\sigma_3 - \sigma_1)(\theta_t - \sigma_1) + \gamma(-1 - \kappa_1)\frac{\zeta_t - \kappa_1}{1 + \kappa_1} &= 0, \quad \textit{for } t \le \tau.
\end{aligned}$$

This leads to the following result.

Existence of a Martingale Measure

Proposition

The pair (θ, ζ) *satisfies the following equations, for* $t \leq \tau$,

$$\theta_t a = \sigma_1 a + c, \quad \zeta_t \gamma a = \kappa_1 \gamma a - (1 + \kappa_1) b.$$

Moreover, for $t > \tau$,

$$\mu_2 - \mu_1 + (\sigma_2 - \sigma_1)(\theta_t - \sigma_1) = 0.$$

Let $a \neq 0$, $\sigma_1 \neq \sigma_2$ *and* $\gamma > b/a$. *Then the unique solution is*

$$\theta_t = \mathbb{1}_{\{t \leq \tau\}} \left(\sigma_1 + \frac{c}{a} \right) + \mathbb{1}_{\{t > \tau\}} \left(\sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} \right), \ \zeta_t = \kappa_1 - \frac{(1 + \kappa_1) b}{\gamma a} > -1.$$

The model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ *is arbitrage-free, complete, and has the Markov property under the unique martingale measure* $\widetilde{\mathbb{Q}}$.

Example B: Extension of the Black-Scholes Model

- Assume that the asset Y^1 is risk-free, the asset $Y^2 \neq Y^1$ is default-free, and Y^3 is a defaultable asset with zero recovery, so that

$$\begin{aligned} dY_t^1 &= rY_t^1 \, dt, \\ dY_t^2 &= Y_t^2 \left(\mu_2 \, dt + \sigma_2 \, dW_t \right), \\ dY_t^3 &= Y_{t-}^3 \left(\mu_3 \, dt + \sigma_3 \, dW_t - \, dM_t \right). \end{aligned}$$

- This corresponds to the following conditions:

$$\sigma_1 = \kappa_1 = 0, \ \mu_1 = r, \ \sigma_2 \neq 0, \ \kappa_2 = 0, \ \kappa_3 = -1.$$

Hence $a = -\sigma_2 \neq 0$. Assume, in addition, that

$$\gamma > b/a = r - \mu_3 - \frac{\sigma_3}{\sigma_2}(r - \mu_2).$$

Example B (Continued)

- Then we obtain

$$\theta = \frac{r - \mu_2}{\sigma_2}, \quad \zeta = -\frac{b}{\gamma a} = \frac{1}{\gamma}\left(\mu_3 - r - \frac{\sigma_3}{\sigma_2}(\mu_2 - r)\right) > -1.$$

- Consequently, we have under the unique martingale measure $\widetilde{\mathbb{Q}}$

$$\begin{aligned} dY_t^1 &= rY_t^1\, dt, \\ dY_t^2 &= Y_t^2\big(r\, dt + \sigma_2\, d\widehat{W}_t\big), \\ dY_t^3 &= Y_{t-}^3\big(r\, dt + \sigma_3\, d\widehat{W}_t - \, d\widehat{M}_t\big). \end{aligned}$$

- We do not assume here that $b = 0$; if this holds then $\zeta = 0$, as in Example A.
- In Case B, the risk-neutral default intensity $\widehat{\gamma}$ and the statistical default intensity γ are different, in general,

Case of Stopped Trading

- Suppose that the recovery payoff at the time of default is exogenously specified in terms of some economic factors related to the prices of traded assets (e.g. credit spreads).
- The valuation problem for a defaultable claim is reduced to finding its pre-default value, and it is natural to search for a replicating strategy up to default time only.
- It thus suffices to examine the stopped model in which asset prices and all trading activities are stopped at time τ.
- In this case, we search for a pair (θ, ζ) of real numbers satisfying

$$\begin{aligned} \theta a &= \sigma_1 a + c, \\ \zeta \gamma a &= \kappa_1 \gamma a - (1 + \kappa_1) b. \end{aligned}$$

Case of Stopped Trading

- If $a \neq 0$ then the unique solution (θ, ζ) to the above pair of equations is

$$\theta = \sigma_1 + \frac{c}{a}, \quad \zeta = \kappa_1 - \frac{(1+\kappa_1)b}{\gamma a} > -1,$$

  where the last inequality holds provided that $\gamma > b/a$.
- As expected, in the stopped model, we obtain the unique martingale measure $\widetilde{\mathbb{Q}}$ for any choice of recovery coefficients κ_2 and κ_3.
- In the case of stopped trading, hedging of a contingent claim after the default time τ is not considered.

CASE A: PRICING PDEs AND HEDGING

Contingent Claim

Let us now discuss the PDE approach in a model in which the prices of all three primary assets are non-vanishing.

- It is natural to focus on the case when the market model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is complete and arbitrage-free.
- Therefore, we shall work under the assumptions of part (i) in the proposition on the existence of a martingale measure.
- We are interested in the valuation and hedging of a generic contingent claim with maturity T and the terminal payoff $Y = G(Y^1_T, Y^2_T, Y^3_T, H_T)$.
- The technique derived for this case can be easily applied to a defaultable claim that is subject to a fairly general recovery scheme.

Risk-Neutral Price

- Let $a \neq 0$ and $b = 0$, and let $\widetilde{\mathbb{Q}}$ be the unique martingale measure associated with the numeraire Y^1. Then
$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \mathcal{E}_T(\theta W)\mathcal{E}_T(\zeta M)$$
where θ and ζ are explicitly known.
- If $Y(Y^1_T)^{-1}$ is $\widetilde{\mathbb{Q}}$-integrable then the risk-neutral price of Y equals, for every $t \in [0, T]$,
$$\begin{aligned}\pi_t(Y) &= Y^1_t \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big((Y^1_T)^{-1} Y \,\big|\, \mathcal{G}_t\big)\\ &= Y^1_t \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big((Y^1_T)^{-1} G(Y^1_T, Y^2_T, Y^3_T, H_T) \,\big|\, Y^1_t, Y^2_t, Y^3_t, H_t\big)\end{aligned}$$
where the second equality is a consequence of the Markov property of (Y^1, Y^2, Y^3, H) under $\widetilde{\mathbb{Q}}$.

Pricing PDEs: Case A

Proposition

Let the price processes Y^i, $i = 1, 2, 3$ satisfy

$$dY_t^i = Y_{t-}^i \left(\mu_i \, dt + \sigma_i \, dW_t + \kappa_i \, dM_t \right)$$

with $\kappa_i > -1$ for $i = 1, 2, 3$. Assume that $a \neq 0$ and $b = 0$. Then the risk-neutral price $\pi_t(Y)$ of the claim Y equals

$$\pi_t(Y) = \mathbb{1}_{\{t<\tau\}} C(t, Y_t^1, Y_t^2, Y_t^3, 0) + \mathbb{1}_{\{t \geq \tau\}} C(t, Y_t^1, Y_t^2, Y_t^3, 1)$$

for some function

$$C : [0, T] \times \mathbb{R}_+^3 \times \{0, 1\} \to \mathbb{R}.$$

Assume that for $h = 0$ and $h = 1$ the function $C(\cdot, h) : [0, T] \times \mathbb{R}_+^3 \to \mathbb{R}$ belongs to the class $\mathrm{C}^{1,2}([0, T] \times \mathbb{R}_+^3, \mathbb{R})$.

Pricing PDEs: Case A

Proposition

Then the functions $C(\cdot,0)$ *and* $C(\cdot,1)$ *solve the following PDEs:*

$$\partial_t C(\cdot,0) + \sum_{i=1}^{3} (\alpha - \gamma\kappa_i) y_i \partial_i C(\cdot,0) + \frac{1}{2}\sum_{i,j=1}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C(\cdot,0) - \alpha C(\cdot,0)$$
$$+ \gamma\big[C(t, y_1(1+\kappa_1), y_2(1+\kappa_2), y_3(1+\kappa_3), 1) - C(t, y_1, y_2, y_3, 0)\big] = 0$$

and

$$\partial_t C(\cdot,1) + \alpha \sum_{i=1}^{3} y_i \partial_i C(\cdot,1) + \frac{1}{2}\sum_{i,j=1}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C(\cdot,1) - \alpha C(\cdot,1) = 0$$

where $\alpha = \mu_i + \sigma_i \frac{c}{a}$, *subject to the terminal conditions*

$$C(T, y_1, y_2, y_3, 0) = G(y_1, y_2, y_3, 0), \;\; C(T, y_1, y_2, y_3, 1) = G(y_1, y_2, y_3, 1).$$

Comments

- The valuation problem splits into two pricing PDEs, which are solved recursively.
    - In the first step, we solve the PDE satisfied by the post-default pricing function $C(\cdot, 1)$.
    - Next, we substitute this function into the first PDE, and we solve it for the pre-default pricing function $C(\cdot, 0)$.
- The assumption that we deal with only three primary assets and the coefficients are constant can be easily relaxed, but a general result is too heavy to be stated here.
- Observe that the real-world default intensity γ under $\mathbb{Q}$, rather than the risk-neutral default intensity $\hat{\gamma}$ under $\widetilde{\mathbb{Q}}$, enters the valuation PDE.

Black and Scholes PDE

- We consider the set-up of Example A, with $a \neq 0$ and $b = 0$.
- Let $Y = G(Y_T^2)$ for some function $G : \mathbb{R} \to \mathbb{R}$ such that $Y(Y_T^1)^{-1}$ is $\widetilde{\mathbb{Q}}$-integrable.
- It is possible to show that $\pi_t(Y) = C(t, Y_t^2)$.
- The two valuation PDEs of Proposition A2 reduce to a single PDE

$$\partial_t C + (\mu_2 - \sigma_2\theta) y_2 \partial_2 C + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} C - (\mu_2 - \sigma_2\theta) C = 0$$

  with $\theta = (r - \mu_2)/\sigma_2$.
- After simplifications, we obtain the classic Black and Scholes PDE

$$\partial_t C + r y_2 \partial_2 C + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} C - rC = 0.$$

Trading Strategies

- Recall that $\phi = (\phi^1, \phi^2, \phi^3)$ is a self-financing strategy if the processes ϕ^1, ϕ^2, ϕ^3 are $\mathbb{G}$-predictable and the wealth process
$$V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3$$
satisfies
$$dV_t(\phi) = \phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 + \phi_t^3 \, dY_t^3.$$
- We say that ϕ replicates a contingent claim Y if $V_T(\phi) = Y$. If ϕ is a replicating strategy for a claim Y then, for $t \in [0, T]$,
$$\pi_t(Y) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3.$$
- To find a replicating strategy, we combine the sensitivities of the valuation function C with respect to primary assets with the jump $\Delta C_t = C_t - C_{t-}$ associated with default event.

Hedging with Sensitivities and Jumps

Proposition

Under the present the assumptions, the claim $G(Y_T^1, Y_T^2, Y_T^3, H_T)$ is replicated by $\phi = (\phi^1, \phi^2, \phi^3)$, where the components ϕ^i, $i = 2, 3$, are given in terms of the valuation functions $C(\cdot, 0)$ and $C(\cdot, 1)$:

$$\phi_t^2 = \frac{1}{aY_{t-}^2}\left((\kappa_3 - \kappa_1)\Big(\sum_{i=1}^{3} \sigma_i Y_{t-}^i \partial_i C - \sigma_1 C\Big) - (\sigma_3 - \sigma_1)(\Delta C - \kappa_1 C)\right)$$

$$\phi_t^3 = \frac{1}{aY_{t-}^3}\left((\kappa_2 - \kappa_1)\Big(\sum_{i=1}^{3} \sigma_i Y_{t-}^i \partial_i C - \sigma_1 C\Big) - (\sigma_2 - \sigma_1)(\Delta C - \kappa_1 C)\right)$$

and ϕ^1 equals

$$\phi_t^1 = (Y_t^1)^{-1}\Big(C_t - \sum_{i=2}^{3} \phi_t^i Y_t^i\Big).$$

Example A: Extension of the Black-Scholes Model

- Assume that the asset Y^1 is risk-free, the asset $Y^2 \neq Y^1$ is default-free, and Y^3 is a defaultable asset with non-zero recovery, so that

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t\big) \end{aligned}$$

  with $\sigma_2 \neq 0$ and $\kappa_3 \neq 0$, $\kappa_3 > -1$.
- We may assume, without loss of generality, that C does not depend explicitly on the variable y_1.
- Assume that $a = \sigma_2\kappa_3 \neq 0$ and $\sigma_2(r - \mu_3) = \sigma_3(r - \mu_2)$. The following result combines and adapts previous results to the present situation.

Example A: Pricing PDEs

Corollary

The arbitrage price of a claim $Y = G(Y_T^2, Y_T^3, H_T)$ *can be represented as* $\pi_t(Y) = C(t, Y_t^2, Y_t^3, H_t)$, *where* $C(t, y_2, y_3, 0)$ *satisfies*

$$\partial_t C(\cdot,0) + ry_2\partial_2 C(\cdot,0) + y_3(r - \kappa_3\gamma)\partial_3 C(\cdot,0) - rC(\cdot,0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} C(\cdot,0) + \gamma\big(C(t,y_2,y_3(1+\kappa_3),1) - C(t,y_2,y_3,0)\big) = 0$$

with $C(T, y_2, y_3, 0) = G(y_2, y_3, 0)$, *and* $C(t, y_2, y_3, 1)$ *satisfies*

$$\partial_t C(t,y_2,y_3,1) + ry_2\partial_2 C(t,y_2,y_3,1) + ry_3\partial_3 C(t,y_2,y_3,1) - rC(t,y_2,y_3,1) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} C(t,y_2,y_3,1) = 0$$

with $C(T, y_2, y_3, 1) = G(y_2, y_3, 1)$.

Example A: Hedging

Corollary

The replicating strategy for Y equals $\phi = (\phi^1, \phi^2, \phi^3)$, where

$$\begin{aligned}
\phi_t^1 &= (Y_t^1)^{-1}\left(C_t - \sum_{i=2}^{3}\phi_t^i Y_t^i\right), \\
\phi_t^2 &= \frac{1}{\sigma_2\kappa_3 Y_{t-}^2}\left(\kappa_3\sum_{i=2}^{3}\sigma_i y_i \partial_i C(t, Y_{t-}^2, Y_{t-}^3, H_{t-})\right. \\
&\quad \left. - \sigma_3\big(C(t, Y_{t-}^2, Y_{t-}^3(1+\kappa_3), 1) - C(t, Y_{t-}^2, Y_{t-}^3, 0)\big)\right), \\
\phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3}\big(C(t, Y_{t-}^2, Y_{t-}^3(1+\kappa_3), 1) - C(t, Y_{t-}^2, Y_{t-}^3, 0)\big).
\end{aligned}$$

Example A: Survival Claim

- By a survival claim we mean a claim of the form $Y = \mathbb{1}_{\{\tau > T\}} X$, where an $\mathcal{F}_T$-measurable random variable X represents the promised payoff.
- In other words, a survival claim is a contract with zero recovery in the case of default prior to maturity T.
- We assume that the promised payoff has the form $X = G(Y_T^2, Y_T^3)$, where Y_T^i is the (pre-default) value of the ith asset at time T.
- It is obvious that the pricing function $C(\cdot, 1)$ is now equal to zero, and thus we are only interested in the pre-default pricing function $C(\cdot, 0)$.

Example A: Survival Claim

Corollary

The pre-default pricing function $C(\cdot, 0)$ of a survival claim of the form $Y = \mathbb{1}_{\{\tau > T\}} G(Y_T^2, Y_T^3)$ solves the PDE

$$\partial_t C(\cdot,0) + ry_2 \partial_2 C(\cdot,0) + y_3(r - \kappa_3 \gamma)\partial_3 C(\cdot,0) + \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,0) - (r+\gamma)C(\cdot,0) = 0$$

with $C(T, y_2, y_3, 0) = G(y_2, y_3)$. The components ϕ^2 and ϕ^3 of a replicating strategy ϕ are given by the following expressions

$$\phi_t^2 = \frac{1}{\kappa_3 \sigma_2 Y_{t-}^2}\Big(\kappa_3 \sum_{i=2}^{3} \sigma_i Y_{t-}^i \partial_i C(\cdot,0) - \sigma_3 C(\cdot,0)\Big), \quad \phi_t^3 = -\frac{C(\cdot,0)}{\kappa_3 Y_{t-}^3}.$$

CASE B: PRICING PDEs AND HEDGING

Case B: Defaultable Asset with Zero Recovery

Standing assumptions:

- We now assume that the prices Y^1 and Y^2 are strictly positive, but $\kappa_3 = -1$ so that Y^3 is a defaultable asset with zero recovery.
- Of course, the price Y^3_t vanishes after default, that is, on the set $\{t \geq \tau\}$.
- We assume here that $a \neq 0$ and $\sigma_1 \neq \sigma_2$, but we no longer postulate that $b = 0$.
- We still assume that $\gamma > b/a$, however. Let us denote

$$\alpha_i = \mu_i + \sigma_i \frac{c}{a}, \quad \beta_i = \mu_i - \sigma_i \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}.$$

Valuation PDEs: Case B

Proposition

Let the price processes Y^i, $i = 1, 2, 3$, satisfy

$$dY_t^i = Y_{t-}^i \left(\mu_i dt + \sigma_i \, dW_t + \kappa_i \, dM_t \right)$$

with $\kappa_i > -1$ for $i = 1, 2$ and $\kappa_3 = -1$. Assume that

$$a \neq 0, \ \sigma_1 \neq \sigma_2, \ \gamma > b/a.$$

Consider a contingent claim Y with maturity date T and the terminal payoff $G(Y_T^1, Y_T^2, Y_T^3, H_T)$.

In addition, we postulate that the pricing functions $C(\cdot, 0)$ and $C(\cdot, 1)$ belong to the class $\mathrm{C}^{1,2}([0, T] \times \mathbb{R}_+^3, \mathbb{R})$.

Pricing PDEs: Case B

Proposition

Then the pre-default pricing function $C(t, y_1, y_2, y_3, 0)$ *satisfies the* pre-default PDE

$$\partial_t C(\cdot,0) + \sum_{i=1}^{3}(\alpha_i - \gamma\kappa_i) y_i \partial_i C(\cdot,0) + \frac{1}{2}\sum_{i,j=1}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} C(\cdot,0)$$
$$+ \left(\gamma - \frac{b}{a}\right)\left[C(t, y_1(1+\kappa_1), y_2(1+\kappa_2), 0, 1) - C(t, y_1, y_2, y_3, 0)\right]$$
$$- \left(\alpha_1 + \kappa_1 \frac{b}{a}\right) C(\cdot,0) = 0$$

subject to the terminal condition

$$C(T, y_1, y_2, y_3, 0) = G(y_1, y_2, y_3, 0).$$

Pricing PDEs: Case B

Proposition

The post-default pricing function $C(t, y_1, y_2, 1)$ *solves the* post-default PDE

$$\partial_t C(\cdot, 1) + \sum_{i=1}^{2} \beta_i y_i \partial_i C(\cdot, 1) + \frac{1}{2} \sum_{i,j=1}^{2} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot, 1) - \beta_1 C(\cdot, 1) = 0$$

subject to the terminal condition

$$C(T, y_1, y_2, 1) = G(y_1, y_2, 0, 1).$$

The components of the replicating strategy ϕ are given by the general formulae.

Example B (Continued)

- We assume that the processes Y^1, Y^2, Y^3 satisfy

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t - dM_t\big). \end{aligned}$$

- Let us write $\widehat{r} = r + \widehat{\gamma}$, where

$$\widehat{\gamma} = \gamma(1+\zeta) = \gamma - \frac{b}{a} = \gamma + \mu_3 - r + \frac{\sigma_3}{\sigma_2}(r - \mu_2) > 0$$

  stands for the default intensity under $\widetilde{\mathbb{Q}}$.
- The quantity $\widehat{r}$ is interpreted as the credit-risk adjusted short-term rate.
- Straightforward calculations show that the following corollary is valid.

Example B: Pricing PDEs

Corollary

Assume that $\sigma_1 = \kappa_1 = \kappa_2 = 0$, $\kappa_3 = -1$ *and*

$$\gamma > b/a = r - \mu_3 - \frac{\sigma_3}{\sigma_2}(r - \mu_2).$$

Then $C(\cdot, 0)$ *satisfies the PDE*

$$\partial_t C(t, y_2, y_3, 0) + r y_2 \partial_2 C(t, y_2, y_3, 0) + \widehat{r} y_3 \partial_3 C(t, y_2, y_3, 0) - \widehat{r} C(t, y_2, y_3, 0) \\ + \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3, 0) + \widehat{\gamma} C(t, y_2, 1) = 0,$$

with $C(T, y_2, y_3, 0) = G(y_2, y_3, 0)$*, and the function* $C(\cdot, 1)$ *solves*

$$\partial_t C(t, y_2, 1) + r y_2 \partial_2 C(t, y_2, 1) + \frac{1}{2} \sigma_2^2 y_2^2 \partial_{22} C(t, y_2, 1) - r C(t, y_2, 1) = 0,$$

with $C(T, y_2, 1) = G(y_2, 0, 1)$.

Example B: Survival Claim

For a survival claim, we have $C(\cdot, 1) = 0$, and thus we obtain following results.

Corollary

The pre-default pricing function $C(\cdot, 0)$ of a survival claim $Y = \mathbb{1}_{\{\tau > T\}} G(Y_T^2, Y_T^3)$ solves the following PDE:

$$\begin{aligned}\partial_t C(t, y_2, y_3, 0) + r y_2 \partial_2 C(t, y_2, y_3, 0) + \widehat{r} y_3 \partial_3 C(t, y_2, y_3, 0) \\ + \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3, 0) - \widehat{r} C(t, y_2, y_3, 0) = 0\end{aligned}$$

with the terminal condition $C(T, y_2, y_3, 0) = G(y_2, y_3)$.

Corollary B2 (Continued)

Corollary

The components ϕ^2 and ϕ^3 of the replicating strategy are, for every $t < \tau$,

$$\begin{aligned}
\phi^2_t &= \frac{1}{\sigma_2 Y^2_{t-}} \Big(\sum_{i=2}^{3} \sigma_i Y^i_{t-} \partial_i C(t, Y^2_{t-}, Y^3_{t-}, 0) + \sigma_3 C(t, Y^2_{t-}, Y^3_{t-}, 0) \Big), \\
\phi^3_t &= \frac{1}{Y^3_{t-}}\, C(t, Y^2_{t-}, Y^3_{t-}, 0).
\end{aligned}$$

- We have $\phi^3_t Y^3_{t-} = C(t, Y^2_{t-}, Y^3_{t-}, 0)$ for every $t \in [0, T]$. Hence the following relationships holds, for every $t < \tau$,

$$\phi^3_t Y^3_t = C(t, Y^2_t, Y^3_t, 0), \quad \phi^1_t Y^1_t + \phi^2_t Y^2_t = 0.$$

- The last equality is a special case of a balance condition introduced in Bielecki et al. (2006) in a semimartingale set-up.

PDE APPROACH TO BASKET CLAIMS

Case of Two Credit Names

We first consider a special case of two credit names:

- Let τ_1 and τ_2 be strictly positive random variables defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$.
- We introduce the corresponding jump processes $H^i_t = \mathbb{1}_{\{\tau_i \leq t\}}$ for $i = 1, 2$, and we denote by $\mathbb{H}^i$ the filtration generated by the process H^i.
- Finally, we set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2$, where the filtration $\mathbb{F}$ is generated by some Brownian motion W (which is also a $\mathbb{G}$-Brownian motion).
- We now need at least four traded assets, since we deal with three (possibly independent) sources of uncertainty.

Dynamics of Traded Assets

Standing assumptions:

- For the sake of simplicity, we assume that $Y_t^1 = 1$, so that Y^1 represents the savings account corresponding to the short-term rate $r = 0$.
- We postulate that the asset price Y^i satisfies, for $i = 2, 3, 4$,

$$dY_t^i = Y_{t-}^i \left(\mu_i \, dt + \sigma_i \, dW_t + \kappa_i \, dM_t^1 + \psi_i \, dM_t^2 \right)$$

  where M^i is the $\mathbb{Q}$-martingale associated with the default process H^i, that is,

$$M_t^i = H_t^i - \int_0^t \gamma_u^i (1 - H_u^i) \, du.$$

- To ensure the Markov property, we assume that $\gamma_u^i = g_i(u, H_u^1, H_u^2)$.
- Defaults cannot occur simultaneously: $\Delta H_t^1 \Delta H_t^2 = 0$.

Contingent Claim

- Consider a contingent claim of the form

$$Y = G(Y_T^2, Y_T^3, Y_T^4, H_T^1, H_T^2).$$

- Its arbitrage price can be represented as a function

$$\pi_t(Y) = C(t, Y_t^2, Y_t^3, Y_t^4, H_t^1, H_t^2)$$

  or equivalently, as a quadruplet of functions: $C(\cdot, 1, 1)$, $C(\cdot, 0, 1)$, $C(\cdot, 1, 0)$ and $C(\cdot, 0, 0)$.

- The pricing functions satisfy the terminal condition

$$C(T, y_2, y_3, y_4, h_1, h_2) = G(y_2, y_3, y_4, h_1, h_2).$$

  The process $C_t = C(t, Y_t^2, Y_t^3, Y_t^4, H_t^1, H_t^2)$ is a $\mathbb{G}$-martingale under $\widetilde{\mathbb{Q}}$.

Pricing PDEs

Let

- $\widehat{\gamma}_0^1$ and $\widehat{\gamma}_0^2$ be the intensities of τ_1 and τ_2 prior to the first default,
- $\widehat{\gamma}_2^1$ be the intensity of the default time τ_1 on the event $\{\tau_2 \leq t < \tau_1\}$,
- $\widehat{\gamma}_1^2$ be the intensity of the default time τ_2 on the event $\{\tau_1 \leq t < \tau_2\}$.

We obtain the following pricing PDE prior to the first default:

$$\partial_t C(\cdot,0,0) - \sum_{i=2}^{4} (\kappa_i \widehat{\gamma}_0^1 + \psi_i \widehat{\gamma}_0^2) y_i \partial_i C(\cdot,0,0) + \frac{1}{2} \sum_{i,j=2}^{4} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,0,0)$$
$$+ \widehat{\gamma}_0^1 \big(C(\cdot,1,0) - C(\cdot,0,0) \big) + \widehat{\gamma}_0^2 \big(C(\cdot,0,1) - C(\cdot,0,0) \big) = 0.$$

Pricing PDEs (continued)

After the first default, we have

$$\partial_t C(\cdot,1,0) - \sum_{i=2}^{4} \psi_i \widehat{\gamma}_1^2 y_i \partial_i C(\cdot,1,0) + \frac{1}{2}\sum_{i,j=2}^{4} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,1,0)$$
$$+ \widehat{\gamma}_1^2 \big(C(\cdot,1,1) - C(\cdot,1,0) \big) = 0,$$

$$\partial_t C(\cdot,0,1) - \sum_{i=2}^{4} \kappa_i \widehat{\gamma}_2^1 y_i \partial_i C(\cdot,0,1) + \frac{1}{2}\sum_{i,j=2}^{4} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,0,1)$$
$$+ \widehat{\gamma}_2^1 \big(C(\cdot,1,1) - C(\cdot,0,1) \big) = 0,$$

and after the second default

$$\partial_t C(\cdot,1,1) + \frac{1}{2}\sum_{i,j=2}^{4} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot,1,1) = 0.$$

Case of m Credit Names

Standing assumptions:

- Let the random times $\tau_1, \tau_2, \dots, \tau_m$, defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, represent the default times of m credit names.
- Under real-world probability $\mathbb{Q}$, the price processes $Y^1, Y^2, \dots, Y^n$ of primary traded assets are governed by

$$dY_t^i = Y_{t-}^i \Big(\mu_t^i \, dt + \sum_{k=1}^{d} \sigma_i^k(t) \, dW_t^k + \sum_{l=1}^{m} \kappa_i^l(t) \, dM_t^l \Big)$$

where the $\mathbb{G}$-martingales M^l, $l = 1, 2, \dots, m$ are given by

$$M_t^l = H_t^l - \int_0^{\tau_l \wedge t} \gamma_u^l \, du = H_t^l - \int_0^t \xi_u^l \, du.$$

The Markovian Model

- The processes $\mu^i, \sigma_i, \kappa_i$ are given by some functions on $\mathbb{R}_+ \times \mathbb{R}^n$

$$\mu^i_t = \mu_i(t, Y^1_{t-}, \ldots, Y^n_{t-}), \quad \sigma_i(t) = \sigma_i(t, Y^1_{t-}, \ldots, Y^n_{t-})$$

  and

$$\kappa_i(t) = \kappa_i(t, Y^1_{t-}, \ldots, Y^n_{t-}).$$

- The functions above are sufficiently regular, so that the SDE admits a unique strong solution for $i = 1, 2, \ldots, n$.
- The pre-default intensities λ^l are deterministic functions of asset prices, that is,

$$\lambda^l_t = \lambda_l(t, Y^1_{t-}, \ldots, Y^n_{t-})$$

  for every $t \in \mathbb{R}_+$ and $l = 1, 2, \ldots, m$.

Kusuoka's Theorem

Proposition

Any probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ is given by the Radon-Nikodým derivative process η satisfying, for $t \in [0, T]$,

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|\, \mathcal{G}_t = \eta_t = \prod_{k=1}^{d} \mathcal{E}_t\Big(\int_0^{\cdot} \theta_u^k \, dW_u^k \Big) \prod_{l=1}^{m} \mathcal{E}_t\Big(\int_0^{\cdot} \zeta_u^l \, dM_u^l \Big)$$

where $\theta^1, \theta^2, \ldots, \theta^d, \zeta^1, \zeta^2, \ldots, \zeta^m$ are some $\mathbb{G}$-predictable processes such that $\zeta_t^l > -1$ for every $t \in [0, T]$.

The processes $\widetilde{W}^k$, $k = 1, \ldots, d$ and $\widetilde{M}^l$, $l = 1, \ldots, m$ are $\mathbb{G}$-martingales under $\widetilde{\mathbb{Q}}$ where

$$\widetilde{W}_t^k = W_t^k - \int_0^t \theta_u^k \, du, \quad \widetilde{M}_t^l = M_t^l - \int_0^t \xi_u^l \zeta_u^l \, du.$$

Martingale Condition

- Assume that the number of primary traded assets is equal to the number of driving orthogonal martingales $W^1, \ldots, W^d, M^1, \ldots, M^m$ plus one, i.e., $n = d + m + 1$.
- In addition, let the price Y^1 be strictly positive.

Proposition

A probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ is a martingale measure associated with a numeraire Y^1 if and only if the processes θ and ζ satisfy the following equation

$$Y^{i,1}_{t-}\left(\mu_1 - \mu_i + \sum_{k=1}^{d}(\sigma^k_1 - \sigma^k_i)(\theta^k_t - \sigma^k_1) + \sum_{l=1}^{m}\xi^l_t(\kappa^l_1 - \kappa^l_i)\frac{\zeta^l_t - \kappa^l_1}{1 + \kappa^l_1}\right) = 0$$

for $i = 2, 3, \ldots, n$.

Pre-default Martingale Condition

Lemma

Martingale condition can be represented as follows

$$\mathbf{A}_t \mathbf{x}_t = \mathbf{b}_t$$

where:

- $\mathbf{x}_t = (\theta, \lambda\zeta)^T$ *is an* $\mathbb{R}^{d+m}$*-valued process with* $\lambda\zeta = (\lambda^1\zeta^1, \dots, \lambda^m\zeta^m)$,
- *the* $\mathbb{R}^{n-1}$*-valued process* $\mathbf{b}_t$ *is explicitly known,*
- *the* $(n-1) \times (m+d)$ *matrix* $\mathbf{A}_t$ *given by*

$$\mathbf{A}_t = \begin{bmatrix} \sigma_1^1 - \sigma_2^1 & \dots & \sigma_1^d - \sigma_2^d & \frac{\kappa_1^1 - \kappa_2^1}{1+\kappa_1^1} & \dots & \frac{\kappa_1^m - \kappa_2^m}{1+\kappa_1^m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_1^1 - \sigma_n^1 & \dots & \sigma_1^d - \sigma_n^d & \frac{\kappa_1^1 - \kappa_n^1}{1+\kappa_1^1} & \dots & \frac{\kappa_1^m - \kappa_n^m}{1+\kappa_1^m} \end{bmatrix}.$$

Existence of a Martingale Measure

- The pre-default intensities λ_t^l satisfy the equality $\lambda_t^l = \gamma_t^l$ on the event $\{\tau_{(1)} > t\}$, that is, prior to occurrence of the first default.

Proposition

Assume that the pre-default intensities λ_t^l, $l = 1, \ldots, m$ are strictly positive for every $t \in [0, T]$. Then the martingale measure $\widetilde{\mathbb{Q}}$ for the relative prices $Y^{i,1}$, $i = 2, 3, \ldots, m$ stopped at $\tau_{(1)} \wedge T$ exists and is unique if and only if $\mathbf{A}_t^{-1}$ exists.

The Radon-Nikodým derivative of $\widetilde{\mathbb{Q}}$ with respect to $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ is given by

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \prod_{k=1}^{d} \mathcal{E}_T\Big(\int_0^{\cdot} \theta_u^k \, dW_u^k\Big) \prod_{l=1}^{m} \mathcal{E}_T\Big(\int_0^{\cdot} \zeta_u^l \, dM_u^l\Big).$$

First-to-Default Claim (FTDC)

Let us denote $\tau_{(1)} = \tau_1 \wedge \tau_2 \wedge \ldots \wedge \tau_m = \min(\tau_1, \tau_2, \ldots, \tau_m)$.

Definition

A first-to-default claim with maturity T is a defaultable claim $(X, Z, \tau_{(1)})$, where X is a constant amount payable at maturity if no default occurs, and $Z = (Z^1, Z^2, \ldots, Z^l)$ is the vector of $\mathbb{G}$-adapted processes, where $Z^l_{\tau_{(1)}}$ specifies the recovery received at time $\tau_{(1)}$ if the lth name is the first defaulted name, that is, on the event $\{\tau_l = \tau_{(1)} \leq T\}$.

Assumptions:

- The processes Z^l, $l = 1, 2, \ldots, m$, are given by some real-valued functions on $[0, T] \times \mathbb{R}^n$, specifically, $Z^l_t = Z_l(t, Y^1_t, \ldots, Y^n_t)$.
- $X = g(Y^1_T, \ldots, Y^n_T)$ for some function $g : \mathbb{R}^n \to \mathbb{R}$.

Valuation of an FTDC

- Assuming that Y is admissible , that is, $Y(Y^1_{\tau_{(1)}})^{-1}$ is $\widetilde{\mathbb{Q}}$-integrable, we can represent the risk-neutral value of Y on the random interval $[0, \tau_{(1)})$ as follows
$$\pi_t(Y) = Y^1_t \, \mathbb{E}_{\widetilde{\mathbb{Q}}}\big(Y(Y^1_{\tau_{(1)}})^{-1} \,|\, \mathcal{G}_t\big).$$
- In the Markovian set-up, we can deduce the existence of a function $C : [0, T] \times \mathbb{R}^n_+ \to \mathbb{R}$ representing the pre-default price of the claim.

Lemma

There exists a function $C : [0, T] \times \mathbb{R}^n_+ \to \mathbb{R}$ *such that we have for every* $t \in [0, T]$
$$\pi_t(Y) = C(t, Y^1_t, \ldots, Y^n_t)$$
on the event $\{\tau_{(1)} > t\}$.

Pricing PDE for an FTDC

Proposition

The function $C(t, y_1, \dots, y_n)$ satisfies the following PDE

$$\partial_t C + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{d} \sigma_i^k \sigma_j^k y_i y_j \partial_{ij} C + \sum_{i=1}^{n} \Big(\alpha_i - \sum_{l=1}^{m} \kappa_i^l \lambda^l (1 + \zeta^l) \Big) y_i \partial_i C$$
$$- (\alpha_1 + \beta) C + \sum_{l=1}^{m} \lambda^l \frac{1 + \zeta^l}{1 + \kappa_1^l} \Delta_l C = 0$$

with the terminal condition $C(T, y_1, \dots, y_n) = g(y_1, \dots, y_n)$, where

$$\alpha_i = \mu_i + \sum_{k=1}^{d} \sigma_i^k (\theta^k - \sigma_1^k), \quad \beta = \sum_{l=1}^{m} \lambda^l \kappa_1^l \Big(1 - \frac{1 + \zeta^l}{1 + \kappa_1^l} \Big),$$

and

$$\Delta_l C = Z_l(t, y_1(1 + \kappa_1^l), \dots, y_n(1 + \kappa_n^l)) - C(t, y_1, \dots, y_n).$$

Replication of an FTDC

- Let C_t be a candidate for the pre-default arbitrage price of an FTDC $(X, Z, \tau_{(1)})$.
- Our goal is to establish the existence of a self-financing trading strategy ϕ such that

$$C_t = V_t(\phi) = \sum_{i=1}^{n} \phi_t^i Y_t^i$$

on the interval $[0, \tau_{(1)} \wedge T]$.

- Equivalently, $\widetilde{C} = C(Y^1)^{-1}$ satisfies

$$d\widetilde{C}_t = d\left(\frac{V_t(\phi)}{Y_t^1}\right) = \sum_{i=2}^{n} \phi_t^i \, dY_t^{i,1}.$$

- In that case, we say that a trading strategy ϕ replicates an FTDC
- We will show that an FTDC can be replicated and thus the pre-default risk-neutral value is also the arbitrage price of an FTDC prior to default.

Notation

- Let $\mathbf{P}^1_t$ stand for the $1 \times d$ vector

$$\mathbf{P}^1_t = \left[\begin{array}{ccc} \sum_{i=1}^n \sigma^1_i Y^i_{t-} \partial_i C - \sigma^1_1 C_{t-} & \dots & \sum_{i=1}^n \sigma^d_i Y^i_{t-} \partial_i C - \sigma^d_1 C_{t-} \end{array} \right]$$

- Let $\mathbf{P}^2_t$ the $1 \times m$ vector for the $1 \times m$ vector

$$\mathbf{P}^2_t = \left[\begin{array}{ccc} \frac{\Delta_1 C_t - \kappa^1_1 C_{t-}}{1+\kappa^1_1} & \dots & \frac{\Delta_m C_t - \kappa^m_1 C_{t-}}{1+\kappa^m_1} \end{array} \right].$$

Lemma

Lemma

The Itô differential of $\widetilde{C}_t$ can be represented as follows

$$d\widetilde{C}_t = (Y^1_{t-})^{-1}\mathbf{P}_t\, d\widetilde{\mathbf{w}}_t$$

where $\mathbf{P}_t = [\mathbf{P}^1_t, \mathbf{P}^2_t]$ *and*

$$d\widetilde{w}_t = \begin{bmatrix} d\widetilde{W}^1_t \\ \vdots \\ d\widetilde{W}^d_t \\ d\widetilde{M}^1_t \\ \vdots \\ d\widetilde{M}^m_t \end{bmatrix}.$$

Lemma

Lemma

The joint dynamics of relative prices $Y_t^{i,1}$, $i = 2, \ldots, n$ *can be represented as follows*

$$d\mathbf{y}_t = \mathbf{Y}_{t-} \mathbf{A}_t \, d\widetilde{\mathbf{w}}_t$$

where $\mathbf{y}_t$ *is the* $(n-1) \times 1$ *vector*

$$\mathbf{y}_t = \begin{bmatrix} Y_t^{2,1} \\ \vdots \\ Y_t^{n,1} \end{bmatrix}$$

and the diagonal $(n-1) \times (n-1)$ *matrix* $\mathbf{Y}_{t-}$ *equals*

$$\mathbf{Y}_{t-} = \begin{bmatrix} Y_{t-}^{2,1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & Y_{t-}^{n,1} \end{bmatrix}.$$

Replicating Strategy

Proposition

Consider a first-to-default claim $(X, Z, \tau_{(1)})$ with the pricing function C. The claim can be replicated by the self-financing trading strategy $\phi = (\phi^1, \ldots, \phi^n)$ where

$$(\phi_t^2, \ldots, \phi_t^n) = (Y_{t-}^1)^{-1} \mathbf{P}_t \mathbf{Y}_t^{-1} \mathbf{A}_t^{-1}$$

and

$$\phi_t^1 = (Y_t^1)^{-1} \Big(C_t - \sum_{i=2}^{n} \phi_t^i Y_t^i \Big).$$

Example: Four Assets and Two Defaults

- We consider a market model with four primary assets that are driven by two possible sources of default and a one-dimensional Brownian motion. We thus have under the real-world probability $\mathbb{Q}$, for $i = 1, \ldots, 4$,

$$dY_t^i = Y_{t-}^i \Big(\mu_i(t)\, dt + \sigma_i^1(t)\, dW_t^1 + \sum_{l=1}^{2} \kappa_i^l(t)\, dM_t^l \Big).$$

- Note that condition $n = m + d + 1$ is satisfied and the matrix $\mathbf{A}_t$ becomes

$$\mathbf{A}_t = \begin{bmatrix} \sigma_1^1 - \sigma_2^1 & \frac{\kappa_1^1 - \kappa_2^1}{1+\kappa_1^1} & \frac{\kappa_1^2 - \kappa_2^2}{1+\kappa_1^2} \\ \sigma_1^1 - \sigma_3^1 & \frac{\kappa_1^1 - \kappa_3^1}{1+\kappa_1^1} & \frac{\kappa_1^2 - \kappa_3^2}{1+\kappa_1^2} \\ \sigma_1^1 - \sigma_4^1 & \frac{\kappa_1^1 - \kappa_4^1}{1+\kappa_1^1} & \frac{\kappa_1^2 - \kappa_4^2}{1+\kappa_1^2} \end{bmatrix}.$$

Example (continued)

- Assuming that the matrix $\mathbf{A}_t$ is non-singular and $\lambda^l_t \neq 0$ for $t \in [0, T]$, we find that the unique martingale measure $\widetilde{\mathbb{Q}}$ is given by

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \mathcal{E}_T\Big(\int_0^{\cdot} \theta^1_u \, dW^1_u\Big) \prod_{l=1}^{2} \mathcal{E}_T\Big(\int_0^{\cdot} \zeta^l_u \, dM^l_u\Big)$$

where θ^1, ζ^1 and ζ^2 are given by

$$\begin{bmatrix} \theta^1 \\ \lambda^1 \zeta^1 \\ \lambda^2 \zeta^2 \end{bmatrix} = \mathbf{A}_t^{-1} \mathbf{b}_t$$

with

$$\mathbf{b}_t = \begin{bmatrix} \mu_2 - \mu_1 + \sigma^1_1(\sigma^1_1 - \sigma^1_2) + \sum_{l=1}^2 \lambda^l(\kappa^l_1 - \kappa^l_2)\frac{\kappa^l_1}{1+\kappa^l_1} \\ \mu_3 - \mu_1 + \sigma^1_1(\sigma^1_1 - \sigma^1_3) + \sum_{l=1}^2 \lambda^l(\kappa^l_1 - \kappa^l_3)\frac{\kappa^l_1}{1+\kappa^l_1} \\ \mu_4 - \mu_1 + \sigma^1_1(\sigma^1_1 - \sigma^1_4) + \sum_{l=1}^2 \lambda^l(\kappa^l_1 - \kappa^l_4)\frac{\kappa^l_1}{1+\kappa^l_1} \end{bmatrix}.$$

Example (continued)

- The dynamics of relative prices $Y^{i,1}$, $i = 2, 3, 4$, under $\widetilde{\mathbb{Q}}$ are given by

$$dY_t^{i,1} = Y_{t-}^{i,1}\Big((\sigma_i^1 - \sigma_1^1)\, d\widetilde{W}_t^1 - \sum_{l=1}^{2} \frac{\kappa_i^l - \kappa_1^l}{1+\kappa_1^l}\, d\widetilde{M}_t^l\Big).$$

  Consider a first-to-default claim $(X, Z, \tau_{(1)})$ where $Z = (Z^1, Z^2)$. Then $\mathbf{P}_t$ becomes

$$\mathbf{P}_t = \Big[\ \textstyle\sum_{i=1}^4 \sigma_i^1 Y_{t-}^i \partial_i C - \sigma_1^1 C_{t-} \quad \frac{\Delta_1 C_t - \kappa_1^1 C_{t-}}{1+\kappa_1^1} \quad \frac{\Delta_2 C_t - \kappa_1^2 C_{t-}}{1+\kappa_1^2}\ \Big]$$

  where the function C solves the pre-default pricing PDE
- The replicating strategy for an FTDC $(X, Z, \tau_{(1)})$ can be found from the equality

$$(\phi_t^2, \phi_t^3, \phi_t^4) = (Y_{t-}^1)^{-1} \mathbf{P}_t \mathbf{Y}_t^{-1} \mathbf{A}_t^{-1},$$

  combined with the formula

$$\phi_t^1 = (Y_t^1)^{-1}\Big(C_t - \sum_{i=2}^{4} \phi_t^i Y_t^i\Big).$$

Final Remarks

In a single-name case:

- we distinguished between the case of strictly positive assets and the case of zero recovery for defaultable asset,
- we examined the pre-default and post-default pricing PDEs,
- explicit representation for replicating strategies were derived.

In a multi-name case:

- we concentrated on the case of a first-to-default claim,
- the pricing PDE and the formula for replicating strategy were derived,
- the method can be extended to kth-to-default claims.