PDE APPROACH TO THE VALUATION AND HEDGING OF BASKET CREDIT DERIVATIVES

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Contents

1 Introduction

The goal of this paper is to examine the PDE approach to the valuation and hedging of defaultable claims in a Markovian model of credit risk. Our approach is largely based on the previous work by Bielecki et al. [3] (for related results, see also [4, 5]). In contrast to [3], however, we consider here a much more general situation, in the sense that the number of primary traded assets, the dimension of the driving Brownian motion, as well as the number of default times are a priori taken to be arbitrary integers. The main results of this note, Propositions 4.1 and 4.3, cover the corresponding results established in [3] (see Propositions 3.1 and 3.2 therein) as special cases.

The paper is organized as follows. For the reader's convenience, we give in Section 2 an overview of relevant definitions of stochastic default intensities. In particular, in Definition 2.8 we deal with the so-called pre-default intensities. In Section 3, we introduce a Markovian security market model and we study its arbitrage-free property in terms of the existence of a martingale measure for relative prices.

Section 4 is devoted to the main issue – the valuation and hedging of first-to-default credit derivatives through the PDE approach. The work concludes with few examples in which we find an explicit representation for the unique martingale measure for a market model and we derive closed-form expressions for replicating strategies of a first-to-default claim.

2 Default Times and Stochastic Intensities

In this introductory section, we provide an overview of the basic properties of default times and the associated stochastic intensities. For more details and proofs, we refer to Bielecki and Rutkowski [2] (see also [1, 6]).

Let the random times $\tau_1, \ldots, \tau_m$, defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{P})$ where $\mathbb{P}$ is the real-world probability, represent the *default times* of m reference credit names. We denote by $\tau_{(1)} = \tau_1 \wedge \ldots \wedge \tau_m = \min(\tau_1, \ldots, \tau_m)$ the moment of the first default, so that, for any $t \in \mathbb{R}_+$, no defaults are observed on the event $\{\tau_{(1)} > t\}$.

Let

$$F(t_1, \ldots, t_n) = \mathbb{P}\{\tau_1 \leq t_1, \ldots, \tau_m \leq t_m\}$$

be the joint probability distribution function of default times. We assume that the probability distribution of default times admits the joint probability density function $f(t_1, \ldots, t_m)$. Also, let

$$G(t_1, \ldots, t_m) = \mathbb{P}\{\tau_1 > t_1, \ldots, \tau_m > t_m\}$$

stand for the joint probability that the names $1, \ldots, m$ have survived up to times $t_1, \ldots, t_m$ respectively. In particular, the *joint survival function* equals

$$\mathbb{P}(\tau_{(1)} > t) = G_{(1)}(t) = G(t, \ldots, t) = \mathbb{P}\{\tau_1 > t, \ldots, \tau_m > t\}.$$

Obviously, $G_{(1)}(t)$ is the probability that no default have occurred prior to time t.

Definition 2.1 For any $l = 1, \ldots, m$, we define the *default indicator process* $H_t^l = \mathbb{1}_{\{\tau_l \leq t\}}$ of the lth credit name and we denote by $\mathbb{H}^l$ the filtration generated by this process, that is, we set $\mathbb{H}^l = (\mathcal{H}_t^l)_{t \in \mathbb{R}_+}$ where $\mathcal{H}_t^l = \sigma(H_u^l : u \leq t)$.

In words, the σ-field $\mathcal{H}_t^l$ represents all of the information gained from observing the default process of the lth credit name up to time t.

We denote by $\mathbb{H}$ the joint filtration generated by default indicator processes $H^1, \ldots, H^m$, so that $\mathbb{H} = \mathbb{H}^1 \vee \ldots \vee \mathbb{H}^m$. Put equivalently, the σ-field $\mathcal{H}_t$ equals $\mathcal{H}_t = \mathcal{H}_t^1 \vee \ldots \vee \mathcal{H}_t^m = \sigma(\mathcal{H}_t^1, \ldots, \mathcal{H}_t^m)$ for any $t \in \mathbb{R}_+$.

Also, we write $H_t^{(1)} = \mathbb{1}_{\{\tau_{(1)} \leq t\}}$ and $\mathbb{H}^{(1)} = (\mathcal{H}_t^{(1)})_{t \in \mathbb{R}_+}$ where $\mathcal{H}_t^{(1)} = \sigma(H_u^{(1)} : u \leq t)$. It is clear that $\mathbb{H}^{(1)}$ is a sub-filtration of $\mathbb{H}$ and thus $\tau_{(1)}$ is an $\mathbb{H}$-stopping time.

Assumption 2.1 We assume that $\mathbb{P}\{\tau_{(1)} > t\} = G_{(1)}(t) > 0$ for every $t \in \mathbb{R}_+$. Moreover, we assume that $\mathbb{P}\{\tau_l = \tau_j\} = 0$ for any $l \neq j$, $l, j = 1, \ldots, m$, so that

$$H_t^{(1)} = H_{t \wedge \tau_{(1)}}^{(1)} = \sum_{l=1}^{m} H_{t \wedge \tau_{(1)}}^l.$$

Finally, we introduce a *reference filtration* $\mathbb{F}$ where $\mathcal{F}_t \subseteq \mathcal{G}$ for every $t \in \mathbb{R}_+$.

Definition 2.2 The *full filtration* $\mathbb{G}$ is defined by setting $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \ldots \vee \mathbb{H}^m$. Equivalently, the σ-field $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^1 \vee \ldots \vee \mathcal{H}_t^m$ is the σ-field generated by the union of the σ-field $\mathcal{F}_t$ and $\mathcal{H}_t$.

In words, the σ-field $\mathcal{G}_t$ represents all of the information gained up to time t from combining our observations of defaults of m credit names with those of price fluctuations due to the underlying noise process (typically, a Brownian motion).

We will sometimes assume, for simplicity of exposition, that $\mathbb{F} = \mathbb{F}^0$ is the trivial filtration. In that case, we will have $\mathbb{G} = \mathbb{H}$. The next definition introduces some additional notation for sub-filtrations of $\mathbb{G}$.

Definition 2.3 Let $\widehat{\mathbb{G}}^l$ stand the filtration generated by the filtrations $\mathbb{F}$ and $\mathbb{H}^l$, so that $\widehat{\mathbb{G}}^l = \mathbb{F} \vee \mathbb{H}^l$. We denote by $\mathbb{G}^l$ the filtration given by

$$\mathbb{G}^l = \mathbb{F} \vee \mathbb{H}^1 \vee \ldots \vee \mathbb{H}^{l-1} \vee \mathbb{H}^{l+1} \vee \ldots \vee \mathbb{H}^m.$$

Hence the full filtration $\mathbb{G}$ satisfies $\mathbb{G} = \mathbb{G}^l \vee \mathbb{H}^l = \mathbb{G}^l \vee \widehat{\mathbb{G}}^l$.

2.1 Marginal Default Intensities

In the first step, we introduce default intensities associated with the marginal distributions of default times and the reference filtration $\mathbb{F}$.

Definition 2.4 We set $\widehat{F}^l_t = \mathbb{P}\{\tau_l \leq t \,|\, \mathcal{F}_t\}$ and we define the $\mathbb{F}$-*survival process* $\widehat{G}^l$ for the lth credit name by the formula

$$\widehat{G}^l_t = 1 - \widehat{F}^l_t = \mathbb{P}\{\tau_l > t \,|\, \mathcal{F}_t\}, \quad \forall\, t \in \mathbb{R}_+.$$

Assume that $\widehat{G}^l_t > 0$, $t \in \mathbb{R}_+$, for every $l = 1, \ldots, m$. Then the $\mathbb{F}$-*hazard process* $\widehat{\Gamma}^l$ of τ_l is defined through the equality

$$1 - \widehat{F}^l_t = e^{-\widehat{\Gamma}^l_t} = -\ln \widehat{G}^l_t, \quad \forall\, t \in \mathbb{R}_+.$$

Assumption 2.2 We assume that $\widehat{G}^l_t > 0$, $t \in \mathbb{R}_+$, and that the process $\widehat{F}^l$ is absolutely continuous with respect to the Lebesgue measure, i.e., $\widehat{F}^l_t = \int_0^t \widehat{f}^l_u \, du$ for some $\mathbb{F}$-predictable, non-negative process $\widehat{f}^l$, for every $l = 1, \ldots, m$.

The last assumption implies that $\widehat{\Gamma}^l$ is also absolutely continuous with respect to the Lebesgue measure, for every $l = 1, \ldots, m$. Specifically, we can express $\widehat{\Gamma}^l_t$ as $\widehat{\Gamma}^l_t = \int_0^t \widehat{\gamma}^l_u \, du$ for the $\mathbb{F}$-predictable, non-negative process $\widehat{\gamma}^l$ given as

$$\widehat{\gamma}^l_t = \frac{\widehat{f}^l_t}{1 - \widehat{F}^l_t} = \frac{\widehat{f}^l_t}{\widehat{G}^l_t}.$$

Definition 2.5 The process $\widehat{\gamma}^l$ is called the $\mathbb{F}$-*intensity* of default time τ_l or, less formally, the *marginal default intensity* of the lth credit name.

The intuitive interpretation of the marginal intensity $\widehat{\gamma}^l$ can be seen from the following convergence, which can be established under mild technical assumptions,

$$\begin{aligned}
\widehat{\gamma}^l_t &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}\{t < \tau_l < t + \Delta t \,|\, \tau_l > t, \mathcal{F}_t\} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\mathbb{P}\{t < \tau_l < t + \Delta t \,|\, \mathcal{F}_t\}}{\mathbb{P}\{\tau_l > t \,|\, \mathcal{F}_t\}} = \frac{\widehat{f}^l_t}{1 - \widehat{F}^l_t}.
\end{aligned}$$

The following auxiliary result is well known (see, e.g., Proposition 5.1.3 in Bielecki and Rutkowski [2]).

Lemma 2.1 *Under Assumption 2.2, the process $\widehat{M}^l$ defined by the formula $\widehat{M}^l_t = H^l_t - \widehat{\Gamma}^l_{\tau_l \wedge t}$, $t \in \mathbb{R}_+$, is a $\widehat{\mathbb{G}}^l$-martingale.*

Let us write $\widehat{\xi}^l_t = \widehat{\gamma}^l_t \mathbb{1}_{\{\tau_l > t\}}$. Then we also have that

$$\widehat{M}^l_t = H^l_t - \int_0^{\tau_l \wedge t} \widehat{\gamma}^l_u \, du = H^l_t - \int_0^t \widehat{\xi}^l_u \, du. \tag{1}$$

The process $\widehat{M}^l$ is the compensated martingale process arising in the Doob-Meyer decomposition with respect to $\widehat{\mathbb{G}}^l$ of the default process H^l (note that H^l is a bounded $\widehat{\mathbb{G}}^l$-submartingale).

2.2 Joint Default Intensities

We now introduce that joint default intensity for the lth credit name, that is, the intensity process associated with the reference filtration $\mathbb{F}$ and default indicator processes $H^1, \dots, H^{l-1}, H^{l+1}, \dots, H^m$.

Definition 2.6 Let us set $F^l_t = \mathbb{P}\{\tau_l \leq t \,|\, \mathcal{G}^l_t\}$ and let us define the $\mathbb{G}^l$-*survival process* by the formula

$$G^l_t = 1 - F^l_t = \mathbb{P}\{\tau_l > t \,|\, \mathcal{G}^l_t\}, \quad \forall \, t \in \mathbb{R}_+.$$

Assume that $G^l_t > 0$, $t \in \mathbb{R}_+$, for every $l = 1, \dots, m$. Then the $\mathbb{G}^l$-*hazard process* Γ^l associated with τ_l is defined through the equality

$$1 - F^l_t = e^{-\Gamma^l t} = - \ln G^l_t, \quad \forall \, t \in \mathbb{R}_+.$$

Assumption 2.3 We assume that $G^l_t > 0$, $t \in \mathbb{R}_+$, and the process F^l is absolutely continuous with respect to the Lebesgue measure, that is, $F^l_t = \int_0^t f^l_u \, du$ for some $\mathbb{G}^l$-predictable process f^l, for every $l = 1, \dots, m$.

Under Assumption 2.3, the process Γ^l is absolutely continuous, for every $l = 1, \dots, m$, and we can express Γ^l_t as $\Gamma^l_t = \int_0^t \gamma^l_u \, du$ for the $\mathbb{G}^l$-predictable, non-negative process γ^l given as

$$\gamma^l_t = \frac{f^l_t}{1 - F^l_t} = \frac{f^l_t}{G^l_t}.$$

Definition 2.7 The process γ^l is called the $\mathbb{G}^l$-*intensity* of default time τ_l or, less formally, the *joint default intensity* of the lth credit name.

The interpretation of the joint default intensity γ^l of the lth credit name can be seen from the following a.e. convergence

$$\begin{aligned}
\gamma^l_t &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}\{t < \tau_l < t + \Delta t \,|\, \tau_l > t, \mathcal{G}^l_t\} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\mathbb{P}\{t < \tau_l < t + \Delta t \,|\, \mathcal{G}^l_t\}}{\mathbb{P}\{\tau_l > t \,|\, \mathcal{G}^l_t\}} = \frac{f^l_t}{1 - F^l_t}.
\end{aligned}$$

The next lemma is a counterpart of Lemma 2.1 and thus it also follows from Proposition 5.1.3 in Bielecki and Rutkowski [2].

Lemma 2.2 *Under Assumption 2.3, the process M^l defined by the formula $M^l_t = H^l_t - \Gamma^l_{\tau_l \wedge t}$, $t \in \mathbb{R}_+$, is a $\mathbb{G}$-martingale.*

Let us write $\xi^l_t = \gamma^l_t \mathbb{1}_{\{\tau_l > t\}}$. Then we also have that

$$M^l_t = H^l_t - \int_0^{\tau_l \wedge t} \gamma^l_u \, du = H^l_t - \int_0^t \xi^l_u \, du. \tag{2}$$

The process M^l is the compensated martingale process from the Doob-Meyer decomposition of the default process H^l, which is now considered as a bounded $\mathbb{G}$-submartingale.

2.3 Pre-Default Intensities

Since we will be mainly interested in the valuation and hedging of the so-called *first-to-default claims*, we find it useful to introduce also the concept of pre-default intensities. Recall that τ_l represents the default time of the lth asset, whereas by $\tau_{(l)}$ we denote the moment of the lth default in our model, in particular, $\tau_{(1)} = \min(\tau_1, \ldots, \tau_m)$ is the moment of the first default.

We assume that $\mathbb{P}\{\tau_{(1)} > t \,|\, \mathcal{F}_t\} > 0$ for every $t \in \mathbb{R}_+$. The following definition was introduced in Bielecki et al. [3].

Definition 2.8 The *pre-default intensity* λ is defined by

$$\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}\{t < \tau_{(1)} \leq t + \Delta t \,|\, \tau_{(1)} > t, \mathcal{F}_t\} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\mathbb{P}\{t < \tau_{(1)} < t + \Delta t \,|\, \mathcal{F}_t\}}{\mathbb{P}\{\tau_{(1)} > t \,|\, \mathcal{F}_t\}}.$$

The *pre-default intensity* λ^l of the lth credit name is defined by the formula

$$\lambda^l_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}\{t < \tau_l \leq t + \Delta t \,|\, \tau_{(1)} > t, \mathcal{F}_t\} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{\mathbb{P}\{t < \tau_l < t + \Delta t \,|\, \mathcal{F}_t\}}{\mathbb{P}\{\tau_{(1)} > t \,|\, \mathcal{F}_t\}}.$$

Remark 2.1 Assume that the reference filtration $\mathbb{F}$ is trivial. Then the pre-default intensities $\lambda_l(t)$ are deterministic functions that are, in general, different from the marginal intensities $\widehat{\gamma}^l(t)$. It is also important to note that the equality $\lambda^l_t = \gamma^l_t$ holds on the event $\{\tau_{(1)} > t\}$, that is, prior to occurrence of the first default.

3 Security Market Model

In this section, we introduce a market model and we study its arbitrage-free features by examining the existence and uniqueness of a martingale measure associated with the choice of a particular primary traded asset, with strictly positive price process, as a numeraire (for the general theory, see, e.g., Musiela and Rutkowski [8]).

3.1 Prices of Primary Assets

We shall first specify the dynamics of primary traded assets in our market model. Let n stand for the number of primary traded assets, d for the dimension of the underlying Brownian motion $W = (W^1, \ldots, W^d)$ under the real-world probability $\mathbb{P}$, and m for the number of default times $\tau_1, \ldots, \tau_m$. Note that W is assumed to be a Brownian motion with respect to the filtration $\mathbb{G}$. In fact, it suffices to assume that W is a Brownian motion with respect to $\mathbb{F}$ and then deduce from Assumption 2.3 that it is also a Brownian motion with respect to $\mathbb{G}$. It is also worth stressing that we do not postulate that the equality $m = n$ holds.

Assumption 3.1 We assume that under real-world probability $\mathbb{P}$ the price processes $Y^1, \ldots, Y^n$ of primary traded assets are governed by the following expression

$$dY^i_t = Y^i_{t-}\Big(\tilde{\mu}_i(t)\, dt + \sum_{k=1}^{d} \sigma^k_i(t)\, dW^k_t + \sum_{l=1}^{m} \kappa^l_i(t)\, dH^l_t\Big) \tag{3}$$

where

$$\tilde{\mu}_i(t) = \mu_i(t) - \sum_{l=1}^{m} \kappa^l_i(t)\xi^l_t$$

and μ^i, σ_i^k and some $\kappa_i^l \geq -1$ are $\mathbb{G}$-predictable processes. Equivalently

$$dY_t^i = Y_{t-}^i\Big(\mu_i(t)\,dt + \sum_{k=1}^{d}\sigma_i^k(t)\,dW_t^k + \sum_{l=1}^{m}\kappa_i^l(t)\,dM_t^l\Big)$$

where the $\mathbb{G}$-martingales M^l, $l = 1, \ldots, m$ are given by (cf. (2))

$$M_t^l = H_t^l - \int_0^{\tau_l \wedge t} \gamma_u^l\,du = H_t^l - \int_0^t \xi_u^l\,du.$$

Remark 3.1 Let us recall that we do not allow for the possibility of simultaneous defaults, i.e., $\mathbb{P}\{\tau_l = \tau_j\} = 0$ for $l \neq j$. This implies that

$$\Delta H_t^l \Delta H_t^j = \begin{cases} \Delta H_t^l & \text{if } l = j, \\ 0 & \text{if } l \neq j. \end{cases}$$

Consequently, the quadratic covariation between any two martingales introduced in Lemma 2.2 is zero for $l \neq j$, that is,

$$[M^l, M^j]_t = \sum_{0<u\leq t} \Delta H_u^l \Delta H_u^j = 0, \quad \forall\, t \in \mathbb{R}_+.$$

We also have that $[W^k, M^l] = 0$ for every $k = 1, \ldots, d$ and $l = 1, \ldots, m$, and thus the *driving martingales* $W^1, \ldots, W^d, M^1, \ldots, M^m$ pairwise orthogonal.

We denote by $\sigma_i(t) = (\sigma_i^1(t), \ldots, \sigma_i^d(t))$ the *volatility vector* of the ith asset. Also, we write $\kappa_i(t) = (\kappa_i^1(t), \ldots, \kappa_i^m(t))$. The parameters $\kappa_i^l(t)$ represent *recovery processes* in the sense that the jump $\Delta Y_t^i = Y_t^i - Y_{t-}^i$ at time t in the price process of the ith asset depends on $\kappa_i^l(t)$ through the formula

$$\Delta Y_t^i = \sum_{l=1}^{m} \kappa_i^l(t) Y_{t-}^i \Delta H_t^l$$

or equivalently

$$Y_t^i = \sum_{l=1}^{m} (1 + \kappa_i^l(t)) Y_{t-}^i \Delta H_t^l. \tag{4}$$

In particular, if $\kappa_i^l(t) = 0$ for $l = 1, \ldots, m$ then the ith asset is indifferent with respect to the default risk of all reference credit names. If, on the contrary, we have that $\kappa_i^l(t) = -1$ for every $l = 1, \ldots, m$ then the value of the ith asset necessarily falls to zero at the moment $\tau_{(1)}$ of the first default. Finally, if $\kappa_i^i(t) = -1$ then the ith asset is subject to zero recovery at time τ_i, in the sense that its price falls to zero at τ_i.

3.1.1 Markovian Set-up

For our purposes, it is essential that the considered market model has a Markovian structure. To ensure this property, we make the following standing assumption regarding the model coefficients and pre-default intensities.

Assumption 3.2 The processes $\mu^i, \sigma_i, \kappa_i$ in the SDE (3.1) are given by some functions on $\mathbb{R}_+ \times \mathbb{R}^n$, specifically, $\mu_i(t) = \mu_i(t, Y_{t-}^1, \ldots, Y_{t-}^n)$, $\sigma_i(t) = \sigma_i(t, Y_{t-}^1, \ldots, Y_{t-}^n)$ and $\kappa_i(t) = \kappa_i(t, Y_{t-}^1, \ldots, Y_{t-}^n)$. Moreover, these functions are sufficiently regular, so that the SDE (3.1) admits a unique strong solution for $i = 1, \ldots, n$. In addition, we assume that the pre-default intensities λ^l are deterministic functions of the asset prices, that is, $\lambda_t^l = \lambda_l(t, Y_{t-}^1, \ldots, Y_{t-}^n)$ for every $t \in \mathbb{R}_+$ and $l = 1, \ldots, m$.

Let us observe that the asset prices $(Y^1, \ldots, Y^n)$ are not necessarily Markovian under Assumption 3.2. We have, however, the following result.

Lemma 3.1 *Under Assumption 3.2, the asset prices* $(Y^1,\dots,Y^n)$ *are jointly Markov prior to the moment of the first default.*

Proof. Note that on the random interval $[0,\tau_{(1)})$ (i.e., prior to the first default) the SDE (3) reduces to

$$dY^i_t = Y^i_t\Big(\tilde\mu_i(t)\,dt + \sum_{k=1}^d \sigma^k_i(t)\,dW^k_t\Big) \tag{5}$$

where

$$\tilde\mu_i(t) = \mu_i(t) - \sum_{l=1}^m \kappa^l_i(t)\gamma^l_t = \mu_i(t) - \sum_{l=1}^m \kappa^l_i(t)\lambda^l_t.$$

Hence, under Assumption 3.2, the process $(Y^1,\dots,Y^n)$ is clearly a Markov process prior to the first default. □

3.1.2 Special Case

In the special case where the coefficients μ_i,σ_i,κ_i and λ^l are deterministic functions of time only, the unique solution to (3) can be found explicitly.

Proposition 3.1 *The unique solution* Y^i *to the SDE (3) is given by the formula*

$$\begin{aligned} Y^i_t &= Y^i_0 \exp\Big(\int_0^t \Big(\tilde\mu_i(u) - \frac{1}{2}\sum_{k=1}^d(\sigma^k_i(u))^2\Big)\,du + \sum_{k=1}^d\int_0^t \sigma^k_i(u)\,dW^k_u\Big)\\ &\quad\times \prod_{0<u\le t}\Big(1+\sum_{l=1}^m \kappa^l_i(u)\,\Delta H^l_u\Big). \end{aligned}$$

Proof. This is a well-known result from the theory of SDEs. In particular, we make use of the fact that $\mathbb{P}\{\tau_i=\tau_j\}=0$. □

Note that we have, on the interval $[0,\tau_{(1)})$,

$$\begin{aligned} Y^i_t &= Y^i_0 \exp\Big(\int_0^t \Big(\mu_i(u) - \sum_{l=1}^m \kappa^l_i(u)\lambda^l_u - \frac{1}{2}\sum_{k=1}^d(\sigma^k_i(u))^2\Big)\,du\Big) \\ &\quad\times \exp\Big(\sum_{k=1}^d\int_0^t \sigma^k_i(u)\,dW^k_u\Big). \end{aligned} \tag{6}$$

3.2 Arbitrage-free Property of the Market Model

We take asset Y^1 as the numeraire and we search for a probability measure $\mathbb{Q}$ such that all asset prices expressed in units of the numeraire follow $\mathbb{Q}$-martingales. In order to alleviate notation, we shall omit the variables in coefficients, so that we shall write μ_i rather than $\mu_i(t,Y^1_{t-},\dots,Y^n_{t-})$, etc.

3.2.1 Dynamics of Relative Prices

We assume that $Y^1_0>0$ and $\kappa^l_1>-1$ for $l=1,\dots,m$ so that the inequality $Y^1_t>0$ is valid for every $t\in\mathbb{R}_+$.

Lemma 3.2 *The dynamics of the process* $(Y^1)^{-1}$ *under* $\mathbb{P}$ *are*

$$d\Big(\frac{1}{Y^1_t}\Big) = \frac{1}{Y^1_{t-}}\Big\{\Big(-\mu_1 + \sum_{k=1}^d(\sigma^k_1)^2 + \sum_{l=1}^m \frac{\xi^l_t(\kappa^l_1)^2}{1+\kappa^l_1}\Big)\,dt - \sum_{k=1}^d \sigma^k_1\,dW^k_t - \sum_{l=1}^m \frac{\kappa^l_1}{1+\kappa^l_1}\,dM^l_t\Big\}.$$

Proof. The Itô formula yields

$$d\Big(\frac{1}{Y_t^1}\Big) = \frac{1}{Y_{t-}^1}\Big\{ -\mu_1\,dt - \sum_{k=1}^d \sigma_1^k dW_t^k - \sum_{l=1}^m \kappa_1^l \xi_t^l\,dt + \sum_{k=1}^d (\sigma_1^k)^2\,dt + \sum_{0<u\le t} \Delta\Big(\frac{1}{Y_u^1}\Big)\Big\}.$$

It follow easily from (4) that

$$\Delta\left(\frac{1}{Y_t^1}\right) = \frac{1}{Y_t^1} - \frac{1}{Y_{t-}^1} = \sum_{l=1}^m \frac{-\kappa_1^l}{1+\kappa_1^l}\frac{1}{Y_{t-}^1}\,\Delta H_t^l.$$

To conclude the proof, it suffices to make use of (4). □

Let us define the *relative price* of the ith asset by setting

$$Y_t^{i,1} = Y_t^i (Y_t^1)^{-1}, \quad \forall\, i = 1,\dots,n.$$

Lemma 3.3 *The dynamics of the relative price $Y^{i,1}$ under $\mathbb{P}$ are*

$$\begin{aligned} dY_t^{i,1} &= Y_{t-}^{i,1}\Big\{\Big(\mu_i - \mu_1 - \sum_{k=1}^d \sigma_1^k(\sigma_i^k - \sigma_1^k) - \sum_{l=1}^m \xi_t^l \kappa_1^l \frac{\kappa_i^l - \kappa_1^l}{1+\kappa_1^l}\Big)\,dt \\ &\quad + \sum_{k=1}^d (\sigma_i^k - \sigma_1^k)\,dW_t^k - \sum_{l=1}^m \frac{\kappa_i^l - \kappa_1^l}{1+\kappa_1^l}\,dM_t^l\Big\}. \end{aligned}$$

Proof. The Itô integration by parts formula

$$d(X_t Z_t) = X_{t-}\,dZ_t + Z_{t-}\,dX_t + d[X,Z]_t$$

yields

$$d\left(Y_{t-}^i \frac{1}{Y_t^1}\right) = Y_{t-}^i\, d\Big(\frac{1}{Y_t^1}\Big) + \frac{1}{Y_{t-}^1}\,dY_t^i + d\Big[Y^i, \frac{1}{Y^1}\Big]_t.$$

Using Lemma 3.2, we thus obtain

$$\begin{aligned} dY_t^{i,1} &= Y_{t-}^{i,1}\Big\{\Big(-\mu_1 + \sum_{k=1}^d (\sigma_1^k)^2\Big)\,dt - \sum_{k=1}^d \sigma_1^k\,dW_t^k - \sum_{l=1}^m \frac{\kappa_1^l}{1+\kappa_1^l}\,dH_t^l\Big\} \\ &\quad + Y_{t-}^{i,1}\Big(\mu_i\,dt + \sum_{k=1}^d \sigma_i^k\,dW_t^k + \sum_{l=1}^m \kappa_i^l\,dM_t^l + \sum_{k=1}^d \sigma_i^k(-\sigma_1^k)\,dt\Big) \\ &\quad + d\sum_{u\le t} \Delta Y_u^i \Delta \frac{1}{Y_u^1}. \end{aligned}$$

Observe that

$$\Delta Y_t^i \Delta \frac{1}{Y_t^1} = Y_{t-}^i \frac{1}{Y_{t-}^1} \sum_{l=1}^m \kappa_i^l \Delta H_t^l \frac{-\kappa_1^l}{1+\kappa_1^l}\,\Delta H_t^l = -Y_{t-}^{i,1} \sum_{l=1}^m \frac{\kappa_i^l \kappa_1^l}{1+\kappa_1^l}\,\Delta H_t^l$$

and thus

$$\begin{aligned} dY_t^{i,1} &= Y_{t-}^{i,1}\Big(-\mu_1 + \sum_{k=1}^d \sigma_1^k(\sigma_1^k - \sigma_i^k) + \sum_{l=1}^m \frac{\xi_t^l}{1+\kappa_1^l}\Big)\,dt \\ &\quad - Y_{t-}^{i,1}\Big(\sum_{k=1}^d \sigma_1^k\,dW_t^k - \sum_{l=1}^m \frac{\kappa_1^l}{1+\kappa_1^l}\,dM_t^l\Big) \\ &\quad + Y_{t-}^{i,1}\Big(\mu_i\,dt + \sum_{k=1}^d \sigma_i^k\,dW_t^k + \sum_{l=1}^m \kappa_i^l\,dM_t^l\Big) - Y_{t-}^{i,1} \sum_{l=1}^m \frac{\kappa_i^l \kappa_1^l}{1+\kappa_1^l}\,dH_t^l. \end{aligned}$$

Substituting for dH_t^l (cf. (2)), we obtain

$$
\begin{aligned}
dY_t^{i,1} \;&=\; Y_{t-}^{i,1}\Big[-\mu_1+\sum_{k=1}^d \sigma_1^k(\sigma_1^k-\sigma_i^k)+\sum_{l=1}^m \xi_t^l\Big(\frac{1}{1+\kappa_1^l}-1+\kappa_1^l\Big)\Big]\,dt\\
&\quad + Y_{t-}^{i,1}\Big[-\sum_{l=1}^m \frac{\kappa_1^l}{1+\kappa_1^l}\,dM_t^l+\mu_i\,dt+\sum_{k=1}^d \sigma_i^k\,dW_t^k+\sum_{l=1}^m \kappa_i^l\,dM_t^l\Big]\\
&\quad - Y_{t-}^{i,1}\Big[\sum_{l=1}^m \frac{\kappa_i^l\kappa_1^l}{1+\kappa_1^l}(dM_t^l+\xi_t^l dt)\Big]-Y_{t-}^{i,1}\sum_{k=1}^d \sigma_1^k\,dW_t^k,
\end{aligned}
$$

and finally

$$
\begin{aligned}
dY_t^{i,1} \;&=\; Y_{t-}^{i,1}\Big\{\Big(\mu_i-\mu_1-\sum_{k=1}^d \sigma_1^k(\sigma_i^k-\sigma_1^k)-\sum_{l=1}^m \xi_t^l\kappa_1^l\frac{\kappa_i^l-\kappa_1^l}{1+\kappa_1^l}\Big)\,dt\\
&\quad +\sum_{k=1}^d(\sigma_i^k-\sigma_1^k)\,dW_t^k-\sum_{l=1}^m \kappa_1^l\frac{\kappa_i^l-\kappa_1^l}{1+\kappa_1^l}\,dM_t^l\Big\},
\end{aligned}
$$

which is the desired result. □

3.2.2 Martingale Measure for Relative Prices

We fix a horizon date $T>0$ and we introduce the following definition.

Definition 3.1 A probability measure $\mathbb{Q}$ is called an (equivalent) *martingale measure* associated with the numeraire Y^1 if $\mathbb{Q}$ is equivalent to the real-world probability $\mathbb{P}$ on $(\Omega,\mathcal{G}_T)$ and the relative price $Y_t^{i,1}$, $t\in[0,T]$, is a $\mathbb{G}$-martingale under $\mathbb{Q}$ for any $i=1,\dots,n$.

In order to examine the existence and uniqueness of a martingale measure associated with the numeraire Y^1, we shall use the following version of Girsanov's theorem, due to Kusuoka [7], in which we denote by $\mathcal{E}(M)$ the Doléans (or stochastic) exponential of a martingale M (see, for instance, Protter [9]). Let us stress that this result is valid under our standing Assumption 2.3.

Proposition 3.2 *Any probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G}_T)$ is given by the Radon-Nikodým derivative process η satisfying, for $t\in[0,T]$,*

$$
\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|\,\mathcal{G}_t=\eta_t=\prod_{k=1}^d \mathcal{E}_t\Big(\int_0^{\cdot}\theta_u^k\,dW_u^k\Big)\prod_{l=1}^m \mathcal{E}_t\Big(\int_0^{\cdot}\zeta_u^l\,dM_u^l\Big) \tag{7}
$$

where

$$
d\eta_t=\eta_{t-}\Big(\sum_{k=1}^d \theta_t^k\,dW_t^k+\sum_{l=1}^m \zeta_t^l\,dM_t^l\Big),\qquad \eta_0=1,
$$

and $\theta^1,\theta^2,\dots,\theta^d,\zeta^1,\zeta^2,\dots,\zeta^m$ are some $\mathbb{G}$-predictable processes such that $\zeta_t^l>-1$ for every $t\in[0,T]$. Moreover, the processes $\widetilde{W}^k$, $k=1,\dots,d$ and $\widetilde{M}^l$, $l=1,\dots,m$ given as, for $t\in[0,T]$,

$$
\begin{aligned}
\widetilde{W}_t^k &= W_t^k-\int_0^t \theta_u^k\,du,\\
\widetilde{M}_t^l &= M_t^l-\int_0^t \xi_u^l\zeta_u^l\,du=H_t^l-\int_0^t \xi_u^l(1+\zeta_u^l)\,du=H_t^l-\int_0^t \widetilde{\xi}_u^l\,du,
\end{aligned}
$$

are $\mathbb{G}$-martingales under $\widetilde{\mathbb{P}}$.

Of course, the processes $\theta = (\theta^1, \theta^2, \dots, \theta^d)$ and $\zeta = (\zeta^1, \zeta^2, \dots, \zeta^m)$ need to satisfy suitable integrability conditions that ensure that the right-hand side in (7) is well defined and the equality $\mathbb{E}_{\mathbb{Q}}(\eta_T) = 1$ holds. Note also that since the martingale M^l is stopped at time τ_l, we may and do assume in what follows that the process ζ^l is also stopped at τ_l for any $l = 1, \dots, m$.

The next result provides a necessary and sufficient condition for the martingale property of relative prices $Y^{i,1}$ under some equivalent probability measure $\widetilde{\mathbb{P}}$. Of course, $Y^{1,1}_t = 1$ for $t \in [0,T]$ and thus it is obviously a martingale under any probability measure equivalent to $\mathbb{P}$.

Proposition 3.3 *A probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ is a martingale measure associated with the numeraire Y^1 if and only if the processes θ and ζ satisfy the following equation, for $i = 2, 3, \dots, n$,*

$$Y^{i,1}_{t-}\Big(\mu_1 - \mu_i + \sum_{k=1}^d (\sigma^k_1 - \sigma^k_i)(\theta^k_t - \sigma^k_1) + \sum_{l=1}^m \xi^l_t(\kappa^l_1 - \kappa^l_i)\frac{\zeta^l_t - \kappa^l_1}{1 + \kappa^l_1}\Big) = 0. \tag{8}$$

Proof. It suffices to apply Proposition 3.2 to dynamics of the relative price derived in Lemma 3.3 and to use the fact that $\mathbb{Q}$ is a martingale measure associated with the numeraire Y^1 if and only if the drift term in the dynamics of $Y^{i,1}$ vanishes. □

Corollary 3.1 *Assume that a martingale measure $\mathbb{Q}$ exists. Then the dynamics of the relative price $Y^{i,1}$ under $\mathbb{Q}$ are, for any $i = 1, \dots, n$*

$$dY^{i,1}_t = Y^{i,1}_{t-}\Big(\sum_{k=1}^d (\sigma^k_i - \sigma^k_1)\, d\widetilde{W}^k_t - \sum_{l=1}^m \frac{\kappa^l_i - \kappa^l_1}{1 + \kappa^l_1}\, d\widetilde{M}^l_t\Big).$$

Proof. The result follow by combining Lemma 3.3 with Proposition 3.3. □

3.2.3 Existence and Uniqueness of a Martingale Measure

Let us assume temporarily that $Y^{i,1}_t \neq 0$ for $t \in [0,T]$. Then (8) reduces to, for $i = 2, 3, \dots, n$,

$$\mu_1 - \mu_i + \sum_{k=1}^d (\sigma^k_1 - \sigma^k_i)(\theta^k_t - \sigma^k_1) + \sum_{l=1}^m \xi^l_t(\kappa^l_1 - \kappa^l_i)\frac{\zeta^l_t - \kappa^l_1}{1 + \kappa^l_1} = 0$$

or equivalently

$$\sum_{k=1}^d \theta^k_t(\sigma^k_1 - \sigma^k_i) + \sum_{l=1}^m \zeta^l_t \xi^l_t \frac{\kappa^l_1 - \kappa^l_i}{1 + \kappa^l_1} = \mu_i - \mu_1 + \sum_{k=1}^d \sigma^k_1(\sigma^k_1 - \sigma^k_i) + \sum_{l=1}^m \xi^l_t(\kappa^l_1 - \kappa^l_i)\frac{\kappa^l_1}{1 + \kappa^l_1}.$$

Recall that $\xi^l_t = \gamma^l_t \mathbb{1}_{\{t \leq \tau_l\}}$ and prior to the first default we have that $\gamma^l_t = \lambda^l_t$. We thus have the following lemma, in which we only assume that prices of primary assets do not vanish prior to the first default.

Lemma 3.4 *Assume that relative prices are non-zero prior to the first default, that is, $Y^{i,1}_t \neq 0$ for any $t \in [0,T]$ on the event $\{\tau_{(1)} > t\}$. Then the processes θ and ζ satisfy, for $i = 2, 3, \dots, n$,*

$$\begin{aligned} &\sum_{k=1}^d \theta^k_t(\sigma^k_1 - \sigma^k_i) + \sum_{l=1}^m \zeta^l_t \lambda^l_t \frac{\kappa^l_1 - \kappa^l_i}{1 + \kappa^l_1} \\ &= \mu_i - \mu_1 + \sum_{k=1}^d \sigma^k_1(\sigma^k_1 - \sigma^k_i) + \sum_{l=1}^m \lambda^l_t(\kappa^l_1 - \kappa^l_i)\frac{\kappa^l_1}{1 + \kappa^l_1} \end{aligned} \tag{9}$$

for any $t \in [0,T]$ on the event $\{\tau_{(1)} > t\}$.

It is clear that the assumption that $Y_t^{i,1} \neq 0$ for any $t \in [0,T]$ on the event $\{\tau_{(1)} > t\}$ is not restrictive if asset prices are modelled by (3) and the coefficients $\mu_i, \sigma_i, \kappa_i$ and the intensities λ^l are deterministic functions of the time parameter t only. Indeed, the asset prices are then given by (6) on the interval $[0, \tau_{(1)})$, and thus they are non-zero prior to the first default if their initial values are non-zero.

To cover a general case, we need to impose the following standing assumption.

Assumption 3.3 The prices of primary assets $Y^1, \dots, Y^n$ do not vanish prior to the first default, that is, $Y_t^i \neq 0$ for $i = 1, \dots, n$ on the interval $[0, \tau_{(1)})$.

Remark 3.2 In what follows, we shall focus on the valuation and hedging of a first-to-default claim, so that it will be enough to examine the system of equations (9). Let us observe, however, that after each default the system of equations (8) collapses by one dimension, as one of default times drops out. If we had zero recovery for exactly one asset (that is, $\kappa_i^l = -1$ for exactly one i) then the remaining system between defaults resembles the system above, as one primary asset also drops out. The default intensities will differ, however, as generally there is a different kind of dependence between defaults of surviving names after the first default occurs.

Equations (9) are referred to as *pre-default no-arbitrage equations.* The matrix we can form to represent this equations has with $n-1$ rows and $d+m$ columns. Since we wish to establish existence and uniqueness of a solution (θ, ζ) to the pre-default equations, we make the following assumption.

Assumption 3.4 The number of primary traded assets is equal to the number of driving orthogonal martingales $W^1, \dots, W^d, M^1, \dots, M^m$ plus one, that is, $n = d + m + 1$.

We now need to solve the pre-default equations for the unknown processes θ and ζ. The following result follows directly from Lemma 3.4 and Assumption 3.4.

Lemma 3.5 *Equation (9) can be represented by*

$$\mathbf{A}_t \mathbf{x}_t = \mathbf{b}_t$$

with the $\mathbb{R}^{d+m}$-valued process $\mathbf{x}_t = (\theta, \lambda\zeta)^T$ with $\lambda\zeta = (\lambda^1\zeta^1, \dots, \lambda^m\zeta^m)$, the $\mathbb{R}^{n-1}$-valued process $\mathbf{b}_t$ given by the right-hand side of (9), and the $(n-1) \times (m+d)$ matrix $\mathbf{A}_t$ given by

$$\mathbf{A}_t = \begin{bmatrix} \sigma_1^1 - \sigma_2^1 & \dots & \sigma_1^d - \sigma_2^d & \frac{\kappa_1^1 - \kappa_2^1}{1+\kappa_1^1} & \dots & \frac{\kappa_1^m - \kappa_2^m}{1+\kappa_1^m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_1^1 - \sigma_n^1 & \dots & \sigma_1^d - \sigma_n^d & \frac{\kappa_1^1 - \kappa_n^1}{1+\kappa_1^1} & \dots & \frac{\kappa_1^m - \kappa_n^m}{1+\kappa_1^m} \end{bmatrix}.$$

The pre-default equations (9) admit a unique solution if and only if the matrix $\mathbf{A}_t$ is non-singular, that is, $|\mathbf{A}_t| \neq 0$ for $t \in [0,T]$.

Remark 3.3 Of course, under Assumption 3.4, $\mathbf{A}_t$ is the square matrix. If model coefficients and default intensities are deterministic functions of time then $\mathbf{A}, \mathbf{x}$ and $\mathbf{b}$ are deterministic functions as well. Hence in that case a solution (θ, ζ) will be given by a pair of deterministic functions of time.

We are in a position to state the following result on the existence and uniqueness of a martingale measure for relative prices.

Proposition 3.4 *Assume that the pre-default intensities λ_t^l, $l = 1, \dots, m$ are strictly positive for every $t \in [0,T]$. Then the martingale measure $\mathbb{Q}$ for the relative prices $Y^{i,1}$, $i = 2, 3, \dots, m$ stopped* *at $\tau_{(1)} \wedge T$ exists and is unique if and only if $\mathbf{A}_t^{-1}$ exists. The Radon-Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ is given by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{k=1}^{d} \mathcal{E}_T\Big(\int_0^{\cdot} \theta_u^k \, dW_u^k\Big) \prod_{l=1}^{m} \mathcal{E}_T\Big(\int_0^{\cdot} \zeta_u^l \, dM_u^l\Big).$$

Proof. It is enough to observe that there is a unique solution to the system in Lemma 3.5 and, under the present assumptions, there is one-to-one correspondence between the vectors $\lambda\zeta$ and ζ. The Radon-Nikodým derivative is then given by Kusuoka's result, that is, Proposition 3.2. □

In what follows, we will work under the following assumption.

Assumption 3.5 The pre-default intensities λ_t^l, $l = 1, \ldots, m$ are strictly positive for every $t \in [0, T]$ and the inverse matrix $\mathbf{A}_t^{-1}$ exists for every $t \in [0, T]$. In other words, we postulate that the market model admits the unique martingale measure $\mathbb{Q}$ for relative prices $Y^{i,1}$, $i = 2, 3, \ldots, m$ stopped at $\tau_{(1)} \wedge T$.

3.2.4 Trading Strategies

By a *trading strategy* ϕ we mean any $\mathbb{R}^n$-valued, $\mathbb{G}$-predictable stochastic process $\phi = (\phi^1, \ldots, \phi^n)$. The *wealth process* of a trading strategy ϕ is represented by

$$V_t(\phi) = \sum_{i=1}^{n} \phi_t^i Y_t^i.$$

We say that a strategy ϕ is *self-financing* if its wealth process satisfies the following condition

$$dV_t(\phi) = \sum_{i=1}^{n} \phi_t^i \, dY_t^i.$$

The following auxiliary result is well known.

Lemma 3.6 *Assume that the price process Y^1 is strictly positive and define the relative wealth $\widetilde{V}(\phi) = V(\phi)(Y^1)^{-1}$. A strategy ϕ is self-financing whenever*

$$d\widetilde{V}_t(\phi) = \sum_{i=2}^{n} \phi_t^i \, dY_t^{i,1}. \tag{10}$$

Definition 3.2 We denote by Φ the class of all *admissible* trading strategies, that is, all self-financing trading strategies such that the *relative wealth process* $\widetilde{V}(\phi)$ is a $\mathbb{G}$-martingale under the martingale measure $\mathbb{Q}$.

We make the standard assumption that only admissible trading strategies are allowed. Then, in view of Assumption 3.5, there are no arbitrage opportunities in our market model $\mathcal{M} = (Y^1, \ldots, Y^n, \Phi)$ provided that all trading activities are stopped at $\tau_{(1)} \wedge T$.

4 PDE Approach

Recall that we work under the standing Assumptions 3.1-3.5. In particular, it is assumed that the market $\mathcal{M} = (Y^1, \ldots, Y^n, \Phi)$ is arbitrage-free, specifically, the martingale measure for relative prices $Y^{i,1}$ exists and is unique when we restrict our attention to the random interval $[0, \tau_{(1)} \wedge T]$.

4.1 First-to-Default Claims

Since trading is not allowed after $\tau_{(1)}$, it is natural to focus on *first-to-default claims* only.

Definition 4.1 A *first-to-default claim* (*FTDC*) with maturity T is a defaultable claim $(X, Z, \tau_{(1)})$, where X is a constant amount payable at maturity if no default occurs and $Z = (Z^1, \dots, Z^l)$ is an $\mathbb{R}^l$-valued, $\mathbb{G}$-adapted process, where $Z^l_{\tau_{(1)}}$ specifies the recovery payoff received at time $\tau_{(1)}$ if the lth name is the first defaulted name, that is, on the event $\{\tau_l = \tau_{(1)} \le T\}$.

In order to preserve the Markovian feature of our model, we shall only consider first-to-default claims satisfying the following additional assumption.

Assumption 4.1 Recovery processes Z^l, $l = 1, \dots, m$ are given by some real-valued functions on $[0, T] \times \mathbb{R}^n$, specifically, $Z^l_t = Z_l(t, Y^1_t, \dots, Y^n_t)$. Moreover, $X = g(Y^1_T, \dots, Y^n_T)$ for some function $g : \mathbb{R}^n \to \mathbb{R}$.

It is worth noting that we have, on the event $\{\tau_l = \tau_{(1)} \le T\}$,

$$Z_l(\tau_l, Y^1_{\tau_l}, \dots, Y^n_{\tau_l}) = Z_l\big(\tau_l, (1 + \kappa^l_1(\tau_l))Y^1_{\tau_l-}, \dots, (1 + \kappa^l_n(\tau_l))Y^n_{\tau_l-}\big).$$

The price process of a first-to-default claim will take the form of a $\mathbb{G}$-martingale, stopped at time $\tau_{(1)} \wedge T$. Let Y be a European contingent claim settled at time $\tau_{(1)} \wedge T$. Assuming that $Y(Y^1_{\tau_{(1)} \wedge T})^{-1}$ is $\mathbb{Q}$-integrable, we can represent the risk-neutral value of Y on the random interval $[0, \tau_{(1)} \wedge T]$ as follows

$$\pi_t(Y) = Y^1_t\, \mathbb{E}_{\mathbb{Q}}\big(Y(Y^1_{\tau_{(1)} \wedge T})^{-1} \,|\, \mathcal{G}_t\big). \tag{11}$$

In the present Markovian set-up, there exists a pre-default pricing function $C : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ representing the *pre-default risk-neutral value* of the claim, as shown in the following lemma in which we assume suitable integrability of the claim Y associated with $(X, Z, \tau_{(1)})$.

Lemma 4.1 *There exists a function $C : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that we have, for every $t \in [0, \tau_{(1)} \wedge T)$,*

$$\pi_t(Y) = C(t, Y^1_t, \dots, Y^n_t) =: C_t.$$

Proof. It suffices to observe that a first-to-default claim $(X, Z, \tau_{(1)})$ can be represented as a European claim Y, settled at time $\tau_{(1)} \wedge T$, and given by

$$\begin{aligned} Y \;&=\; \sum_{l=1}^{m} \mathbb{1}_{\{\tau_l = \tau_{(1)} \le T\}} Z_l\big(\tau_l, (1 + \kappa^l_1(\tau_l))Y^1_{\tau_l-}, \dots, (1 + \kappa^l_n(\tau_l))Y^n_{\tau_l-}\big) \\ &+\; \mathbb{1}_{\{\tau_{(1)} > T\}}\, g(Y^1_T, \dots, Y^n_T), \end{aligned} \tag{12}$$

and to use the Markov property established in Lemma 3.1. □

4.2 Pre-default Pricing PDE

We assume from now on that $C_t = C(t, Y^1_t, \dots, Y^n_t)$ for some *regular* function C. By *regular*, we mean that the partial derivatives $\partial_i C = \partial_{y_i} C$ and $\partial_{ij} C = \partial_{y_i} \partial_{y_j} C$ are well-defined continuous functions. We say that an FTDC $(X, Z, \tau_{(1)})$ is *admissible* if the random variable $Y(Y^1_{\tau_{(1)} \wedge T})^{-1}$ is $\mathbb{Q}$-integrable, where Y is given by (12) and the associated pre-default pricing function C is regular.

Proposition 4.1 *The pre-default pricing function $C(t, y_1, \dots, y_n)$ of an admissible FTDC $(X, Z, \tau_{(1)})$ satisfies the following PDE*

$$\begin{aligned}\partial_t C + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{d} \sigma_i^k \sigma_j^k y_i y_j \partial_{ij} C + \sum_{i=1}^{n} \Big(\alpha_i - \sum_{l=1}^{m} \kappa_i^l \lambda^l (1 + \zeta^l) \Big) y_i \partial_i C \\ - (\alpha_1 + \beta) C + \sum_{l=1}^{m} \lambda^l \frac{1 + \zeta^l}{1 + \kappa_1^l} \, \Delta_l C = 0\end{aligned}$$

with the terminal condition $C(T, y_1, \dots, y_n) = g(y_1, \dots, y_n)$, where

$$\alpha_i = \mu_i + \sum_{k=1}^{d} \sigma_i^k (\theta^k - \sigma_1^k), \quad \beta = \sum_{l=1}^{m} \lambda^l \kappa_1^l \Big(1 - \frac{1 + \zeta^l}{1 + \kappa_1^l} \Big),$$

and

$$\Delta_l C = Z_l(t, y_1(1 + \kappa_1^l), \dots, y_n(1 + \kappa_n^l)) - C(t, y_1, \dots, y_n).$$

Proof. Using Itô's formula, we obtain (the arguments $(t, Y_t^1, \dots, Y_t^n)$ are suppressed)

$$\begin{aligned} dC_t &= \partial_t C \, dt + \sum_{i=1}^{n} Y_{t-}^i \partial_i C \Big(\mu_i \, dt + \sum_{k=1}^{d} \sigma_i^k \, dW_t^k \Big) + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{d} \sigma_i^k \sigma_j^k Y_{t-}^i Y_{t-}^j \partial_{ij} C \, dt \\ &\quad + \sum_{l=1}^{m} \Big(\Delta_l C_t - \sum_{i=1}^{n} \kappa_i^l Y_{t-}^i \partial_i C \Big) \big(dM_t^l + \lambda^l \, dt \big) \end{aligned}$$

where $\Delta_l C_t$ is defined by the formula

$$\Delta_l C_t = Z_l(t, (1 + \kappa_1^l) Y_{t-}^1, \dots, (1 + \kappa_n^l) Y_{t-}^n) - C(t, Y_{t-}^1, \dots, Y_{t-}^n).$$

We take Y^1 as the numeraire and we shall use the martingale property of the relative price $\widetilde{C}_t = (Y_t^1)^{-1} C(t, Y_t^1, \dots, Y_t^n)$ under the probability measure $\mathbb{Q}$. Another application of Itô's formula yields

$$d\widetilde{C}_t = d\Big(\frac{C_t}{Y_t^1} \Big) = C_{t-} \, d\Big(\frac{1}{Y_t^1} \Big) + \frac{1}{Y_{t-}^1} \, dC_t + d\Big[C, \frac{1}{Y^1} \Big]_t$$

where in turn

$$d\Big[C, \frac{1}{Y^1} \Big]_t = -\frac{1}{Y_t^1} \sum_{i=1}^{n} \sum_{k=1}^{d} \sigma_1^k \sigma_i^k Y_t^i \partial_i C \, dt - \frac{1}{Y_{t-}^1} \sum_{l=1}^{m} \frac{\kappa_1^l}{1 + \kappa_1^l} \, \Delta_l C_t \Delta H_t^l.$$

In view of Lemma 3.2, we thus have

$$\begin{aligned} d\widetilde{C}_t &= \widetilde{C}_{t-} \Big\{ \Big(- \mu_1 + \sum_{k=1}^{d} (\sigma_1^k)^2 + \sum_{l=1}^{m} \frac{\lambda^l (\kappa_1^l)^2}{1 + \kappa_1^l} \Big) \, dt - \sum_{k=1}^{d} \sigma_1^k dW_t^k - \sum_{l=1}^{m} \frac{\kappa_1^l}{1 + \kappa_1^l} \, dM_t^l \Big\} \\ &\quad + \frac{1}{Y_t^1} \Big\{ \partial_t C + \sum_{i=1}^{n} \mu_i Y_{t-}^i \partial_i C + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{d} \sigma_i^k \sigma_j^k Y_{t-}^i Y_{t-}^j \partial_{ij} C \Big\} \, dt \\ &\quad + \frac{1}{Y_t^1} \sum_{i=1}^{n} \sum_{k=1}^{d} \sigma_i^k Y_{t-}^i \partial_i C \, dW_t^k + \frac{1}{Y_{t-}^1} \sum_{l=1}^{m} \Big(\Delta_l C_t - \sum_{i=1}^{n} \kappa_i^l Y_i \partial_i C \Big) (dM_t^l + \lambda^l \, dt) \\ &\quad - \frac{1}{Y_t^1} \sum_{i=1}^{n} \sum_{k=1}^{d} \sigma_1^k \sigma_i^k Y_t^i \partial_i C \, dt - \frac{1}{Y_{t-}^1} \sum_{l=1}^{m} \frac{\kappa_1^l}{1 + \kappa_1^l} \Delta_l C_t (dM_t^l + \lambda^l \, dt). \end{aligned}$$

Under the martingale measure $\mathbb{Q}$, we obtain

$$\begin{aligned}
d\widetilde{C}_t &= \widetilde{C}_{t-}\Big(-\mu_1+\sum_{k=1}^{d}(\sigma_1^k)^2+\sum_{l=1}^{m}\frac{\lambda^l(\kappa_1^l)^2}{1+\kappa_1^l}\Big)\,dt-\widetilde{C}_{t-}\Big\{\sum_{k=1}^{d}\sigma_1^k d\widetilde{W}_t^k+\sum_{k=1}^{d}\sigma_1^k\theta^k\,dt\\
&\quad+\sum_{l=1}^{m}\frac{\kappa_1^l}{1+\kappa_1^l}\Big(d\widetilde{M}_t^l+\lambda^l\zeta^l\,dt\Big)\Big\}\\
&\quad+\frac{1}{Y_t^1}\Big\{\partial_t C+\sum_{i=1}^{n}\mu_i Y_{t-}^i\partial_i C+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{d}\sigma_i^k\sigma_j^k Y_{t-}^i Y_{t-}^j\partial_{ij}C\Big\}\,dt\\
&\quad+\frac{1}{Y_t^1}\sum_{i=1}^{n}\sum_{k=1}^{d}\sigma_i^k Y_{t-}^i\partial_i C(d\widetilde{W}_t^k+\theta^k\,dt)\\
&\quad+\frac{1}{Y_{t-}^1}\sum_{l=1}^{m}\Big(\Delta_l C_t-\sum_{i=1}^{n}\kappa_i^l Y_i\partial_i C\Big)(d\widetilde{M}_t^l+\lambda^l(1+\zeta^l)\,dt)\\
&\quad-\frac{1}{Y_t^1}\sum_{i=1}^{n}\sum_{k=1}^{d}\sigma_1^k\sigma_i^k Y_{t-}^i\partial_i C\,dt-\frac{1}{Y_{t-}^1}\sum_{l=1}^{m}\frac{\kappa_1^l}{1+\kappa_1^l}\Delta_l C_t\big(d\widetilde{M}_t^l+\lambda^l(1+\zeta^l)\,dt\big).
\end{aligned}$$

Consequently

$$\begin{aligned}
d\widetilde{C}_t &= \widetilde{C}_t\Big(-\mu_1+\sum_{k=1}^{d}(\sigma_1^k)^2+\sum_{l=1}^{m}\frac{\lambda^l(\kappa_1^l)^2}{1+\kappa_1^l}\,dt-\sum_{k=1}^{d}\sigma_1^k\theta^k\,dt-\sum_{l=1}^{m}\frac{\kappa_1^l\lambda^l\zeta^l}{1+\kappa_1^l}\,dt\Big)\\
&\quad+\frac{1}{Y_t^1}\Big(\partial_t C+\sum_{i=1}^{n}\mu_i Y_{t-}^i\partial_i C+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{d}\sigma_i^k\sigma_j^k Y_{t-}^i Y_{t-}^j\partial_{ij}C\Big)\,dt\\
&\quad+\frac{1}{Y_t^1}\sum_{i=1}^{n}\sum_{k=1}^{d}\sigma_i^k Y_{t-}^i\partial_i C\theta^k\,dt+\frac{1}{Y_t^1}\sum_{l=1}^{m}\Big(\Delta_l C_t-\sum_{i=1}^{n}\kappa_i^l Y_i\partial_i C\Big)\lambda^l(1+\zeta^l)\,dt\\
&\quad-\frac{1}{Y_t^1}\sum_{i=1}^{n}\sum_{k=1}^{d}\sigma_1^k\sigma_i^k Y_{t-}^i\partial_i C\,dt-\frac{1}{Y_t^1}\sum_{l=1}^{m}\frac{\kappa_1^l}{1+\kappa_1^l}\Delta_l C_t\lambda^l(1+\zeta^l)\,dt+\mathbb{Q}\text{-martingale}.
\end{aligned}$$

The martingale property of $\widetilde{C}_t$ under $\mathbb{Q}$ thus gives

$$\begin{aligned}
&C_t\Big(-\mu_1+\sum_{k=1}^{d}(\sigma_1^k)^2+\sum_{l=1}^{m}\frac{\lambda^l(\kappa_1^l)^2}{1+\kappa_1^l}\,dt-\sum_{k=1}^{d}\sigma_1^k\theta^k\,dt-\sum_{l=1}^{m}\frac{\kappa_1^l\lambda^l\zeta^l}{1+\kappa_1^l}\,dt\Big)\\
&+\Big(\partial_t C+\sum_{i=1}^{n}\mu_i Y_t^i\partial_i C+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{d}\sigma_i^k\sigma_j^k Y_t^i Y_t^j\partial_{ij}C\Big)\,dt\\
&+\sum_{i=1}^{n}\sum_{k=1}^{d}\sigma_i^k Y_t^i\partial_i C\theta^k\,dt+\sum_{l=1}^{m}\Big(\Delta_l C_t-\sum_{i=1}^{n}\kappa_i^l Y_i\partial_i C\Big)\lambda^l(1+\zeta^l)\,dt\\
&-\sum_{i=1}^{n}\sum_{k=1}^{d}\sigma_1^k\sigma_i^k Y_t^i\partial_i C\,dt-\sum_{l=1}^{m}\frac{\kappa_1^l}{1+\kappa_1^l}\Delta_l C_t\lambda^l(1+\zeta^l)\,dt=0.
\end{aligned}$$

After rearrangement, we obtain the desired PDE satisfied by the pre-default pricing function C. □

4.3 Replication of a First-to-Default Claim

In what follows, we only consider admissible first-to-default claims and we work under the assumptions of Proposition 4.1. Let C_t be a candidate for the arbitrage price of an FTDC $(X,Z,\tau_{(1)})$, as given by the risk-neutral valuation formula (11) with Y given by (12). Our goal is to establish the existence of a self-financing trading strategy ϕ such that

$$C_t = V_t(\phi) = \sum_{i=1}^{n} \phi_t^i Y_t^i \tag{13}$$

on the interval $[0, \tau_{(1)} \wedge T]$. Equivalently, by virtue of Lemma 3.6

$$d\widetilde{C}_t = d\Big(\frac{V_t(\phi)}{Y_t^1}\Big) = \sum_{i=2}^{n} \phi_t^i \, dY_t^{i,1}. \tag{14}$$

In that case, we say that a trading strategy ϕ replicates an FTDC. We will show that any FTDC can be replicated and thus the pre-default risk-neutral value is also the arbitrage price of an FTDC prior to default. Put another way, we will establish completeness of the model, in the sense of the following definition.

Definition 4.2 We say that the market model $\mathcal{M} = (Y^1, \dots, Y^n, \Phi)$ is *complete* if any first-to-default claim $(X, Z, \tau_{(1)})$ can be replicated by continuous trading in primary assets.

Proposition 4.2 *The Itô differential $d\widetilde{C}_t$ can be represented as follows*

$$d\widetilde{C}_t = (Y_{t-}^1)^{-1} \mathbf{P}_t \, d\widetilde{\mathbf{w}}_t \tag{15}$$

where

$$d\widetilde{w}_t = \begin{bmatrix} d\widetilde{W}_t^1 \\ \vdots \\ d\widetilde{W}_t^d \\ d\widetilde{M}_t^1 \\ \vdots \\ d\widetilde{M}_t^m \end{bmatrix}$$

and $\mathbf{P}_t = [\mathbf{P}_t^1, \mathbf{P}_t^2]$ where in turn the $1 \times d$ vector $\mathbf{P}_t^1$ equals

$$\mathbf{P}_t^1 = \begin{bmatrix} \sum_{i=1}^n \sigma_i^1 Y_{t-}^i \partial_i C - \sigma_1^1 C_{t-} & \dots & \sum_{i=1}^n \sigma_i^d Y_{t-}^i \partial_i C - \sigma_1^d C_{t-} \end{bmatrix}$$

and the $1 \times m$ vector $\mathbf{P}_t^2$ is given by

$$\mathbf{P}_t^2 = \begin{bmatrix} \frac{\Delta_1 C_t - \kappa_1^1 C_{t-}}{1+\kappa_1^1} & \dots & \frac{\Delta_m C_t - \kappa_1^m C_{t-}}{1+\kappa_1^m} \end{bmatrix}.$$

Proof. Let $\widetilde{C}_t = C_t (Y_t^1)^{-1}$ be the relative price of the claim. Since the drift term in dynamics of $\widetilde{C}$ under $\mathbb{Q}$ vanishes, we have (see the proof of Proposition 4.1)

$$d\widetilde{C}_t = \frac{1}{Y_{t-}^1} \Big\{ \sum_{k=1}^{d} \Big(\sum_{i=1}^{n} \sigma_i^k Y_{t-}^i \partial_i C - \sigma_1^k C_{t-} \Big) d\widetilde{W}_t^k + \sum_{l=1}^{m} \frac{\Delta_l C_t - \kappa_1^l C_{t-}}{1+\kappa_1^l} \, d\widetilde{M}_t^l \Big\},$$

and this yields (15). $\square$

The next result is an immediate consequence of Corollary 3.1. Recall that the matrix $\mathbf{A}_t$ was defined in Lemma 3.5.

Lemma 4.2 *The joint dynamics of relative prices $Y_t^{i,1}$, $i = 2, \dots, n$ can be represented as follows*

$$d\mathbf{y}_t = \mathbf{Y}_{t-} \mathbf{A}_t \, d\widetilde{\mathbf{w}}_t$$

where $\mathbf{y}_t$ *is the* $(n-1) \times 1$ *vector*

$$\mathbf{y}_t = \begin{bmatrix} Y_t^{2,1} \\ \vdots \\ Y_t^{n,1} \end{bmatrix}$$

and the diagonal $(n-1) \times (n-1)$ *matrix* $\mathbf{Y}_{t-}$ *equals*

$$\mathbf{Y}_{t-} = \begin{bmatrix} Y_{t-}^{2,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & Y_{t-}^{n,1} \end{bmatrix}.$$

Proposition 4.3 *Consider a first-to-default claim* $(X, Z, \tau_{(1)})$ *with the pricing function* C. *The claim can be replicated by the self-financing trading strategy* $\phi = (\phi^1, \dots, \phi^n)$ *where*

$$(\phi_t^2, \dots, \phi_t^n) = (Y_{t-}^1)^{-1} \mathbf{P}_t \mathbf{Y}_t^{-1} \mathbf{A}_t^{-1}$$

and

$$\phi_t^1 = (Y_t^1)^{-1} \Big(C_t - \sum_{i=2}^n \phi_t^i Y_t^i \Big).$$

Proof. Recall that we assumed that $\mathbf{A}_t^{-1}$ exists. We thus have that $d\widetilde{\mathbf{w}}_t = \mathbf{A}_t^{-1} \mathbf{Y}_t^{-1} \, d\mathbf{y}_t$ where $\mathbf{Y}_{t-}^{-1}$ is the inverse of $\mathbf{Y}_{t-}$. Hence we can rewrite equation (15) as follows

$$d\widetilde{C}_t = (Y_{t-}^1)^{-1} \mathbf{P}_t \mathbf{Y}_t^{-1} \mathbf{A}_t^{-1} d\mathbf{y}_t. \tag{16}$$

Let us denote $\widetilde{\phi}_t = (\phi_t^2, \dots, \phi_t^n)$. By combining (14) with (16), we obtain

$$d\widetilde{C}_t = (Y_{t-}^1)^{-1} \mathbf{P}_t \mathbf{Y}_t^{-1} \mathbf{A}_t^{-1} d\mathbf{y}_t = \widetilde{\phi}_t \, d\mathbf{y}_t = \sum_{i=2}^n \phi_t^i \, dY_t^{i,1}.$$

This yields the first equality. The second equality follows from (13). □

4.4 Examples

To provide a better insight into our results, we provide in this section few examples. In all cases considered below, we shall assume that the model parameters are such that the corresponding matrix $\mathbf{A}_t$ is non-singular for every $t \in [0, T]$. Also, we shall postulate the the pre-default intensities are strictly positive.

4.4.1 Four Assets and Two Defaults

We consider a market model with four primary assets that are driven by two possible sources of default and a one-dimensional Brownian motion. We thus have under the real-world probability $\mathbb{P}$, for $i = 1, \dots, 4$,

$$dY_t^i = Y_{t-}^i \Big(\mu_i(t) \, dt + \sigma_i^1(t) \, dW_t^1 + \sum_{l=1}^2 \kappa_i^l(t) \, dM_t^l \Big).$$

Note that condition $n = m + d + 1$ is satisfied and the matrix $\mathbf{A}_t$ becomes

$$\mathbf{A}_t = \begin{bmatrix} \sigma_1^1 - \sigma_2^1 & \frac{\kappa_1^1 - \kappa_2^1}{1+\kappa_1^1} & \frac{\kappa_1^2 - \kappa_2^2}{1+\kappa_1^2} \\ \sigma_1^1 - \sigma_3^1 & \frac{\kappa_1^1 - \kappa_3^1}{1+\kappa_1^1} & \frac{\kappa_1^2 - \kappa_3^2}{1+\kappa_1^2} \\ \sigma_1^1 - \sigma_4^1 & \frac{\kappa_1^1 - \kappa_4^1}{1+\kappa_1^1} & \frac{\kappa_1^2 - \kappa_4^2}{1+\kappa_1^2} \end{bmatrix}.$$

Assuming that the matrix $\mathbf{A}_t$ is non-singular and $\lambda^l_t \neq 0$ for $t \in [0,T]$, we find easily that the unique martingale measure $\mathbb{Q}$, in the sense of Definition 3.1, is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_T\Big(\int_0^\cdot \theta^1_u \, dW^1_u \Big) \prod_{l=1}^2 \mathcal{E}_T\Big(\int_0^\cdot \zeta^l_u \, dM^l_u \Big)$$

where θ^1, ζ^1 and ζ^2 are given by

$$\begin{bmatrix} \theta^1 \\ \lambda^1\zeta^1 \\ \lambda^2\zeta^2 \end{bmatrix} = \mathbf{A}_t^{-1}\mathbf{b}_t$$

with

$$\mathbf{b}_t = \begin{bmatrix} \mu_2 - \mu_1 + \sigma^1_1(\sigma^1_1 - \sigma^1_2) + \sum_{l=1}^2 \lambda^l(\kappa^l_1 - \kappa^l_2)\frac{\kappa^l_1}{1+\kappa^l_1} \\ \mu_3 - \mu_1 + \sigma^1_1(\sigma^1_1 - \sigma^1_3) + \sum_{l=1}^2 \lambda^l(\kappa^l_1 - \kappa^l_3)\frac{\kappa^l_1}{1+\kappa^l_1} \\ \mu_4 - \mu_1 + \sigma^1_1(\sigma^1_1 - \sigma^1_4) + \sum_{l=1}^2 \lambda^l(\kappa^l_1 - \kappa^l_4)\frac{\kappa^l_1}{1+\kappa^l_1} \end{bmatrix}.$$

The dynamics of relative prices $Y^{i,1}, i = 2,3,4$, under $\mathbb{Q}$ are given by Corollary 3.1, that is,

$$dY^{i,1}_t = Y^{i,1}_{t-}\Big((\sigma^1_i - \sigma^1_1)\, d\widetilde{W}^1_t - \sum_{l=1}^2 \frac{\kappa^l_i - \kappa^l_1}{1+\kappa^l_1}\, d\widetilde{M}^l_t \Big).$$

Let us consider a first-to-default claim $(X, Z, \tau_{(1)})$ where $Z = (Z^1, Z^2)$. Then the vector $\mathbf{P}_t$ becomes

$$\mathbf{P}_t = \Big[\; \sum_{i=1}^4 \sigma^1_i Y^i_{t-}\partial_i C - \sigma^1_1 C_{t-} \quad \frac{\Delta_1 C_t - \kappa^1_1 C_{t-}}{1+\kappa^1_1} \quad \frac{\Delta_2 C_t - \kappa^2_1 C_{t-}}{1+\kappa^2_1} \; \Big]$$

where the function C solves the pre-default pricing PDE of Proposition 4.1 (as usual, we implicitly assume that the pre-default pricing function C is sufficiently regular). According to Proposition 4.3, the replicating strategy for an FTDC $(X, Z, \tau_{(1)})$ can be found from the equality

$$(\phi^2_t, \phi^3_t, \phi^4_t) = (Y^1_{t-})^{-1}\mathbf{P}_t \mathbf{Y}_t^{-1}\mathbf{A}_t^{-1},$$

combined with the formula

$$\phi^1_t = (Y^1_t)^{-1}\Big(C_t - \sum_{i=2}^4 \phi^i_t Y^i_t \Big).$$

4.4.2 Three Assets and Two Defaults

This example will serve to highlight the effects of defaults. We consider a model with three primary assets, none of which is dependent upon Brownian motion. Thus the assets dynamics contain only drift and the possibility of default. Specifically, for $i = 1,2,3$,

$$dY^i_t = Y^i_{t-}\Big(\mu_i(t)\, dt + \sum_{l=1}^2 \kappa^l_i(t)\, dM^l_t \Big).$$

By applying Proposition 4.1, we obtain the following pricing PDE

$$\partial_t C + \sum_{i=1}^3 \Big(\mu_i - \sum_{l=1}^2 \kappa^l_i \lambda^l (1+\zeta^l) \Big) y_i \partial_i C - (\mu_1 + \beta) C + \sum_{l=1}^2 \lambda^l \frac{1+\zeta^l}{1+\kappa^l_1}\, \Delta_l C = 0$$

where

$$\beta = \sum_{l=1}^2 \lambda^l \kappa^l_1 \Big(1 - \frac{1+\zeta^l}{1+\kappa^l_1} \Big)$$

and, for $l = 1, 2$,

$$\Delta_l C = Z_l(t, y_1(1+\kappa_1^l), \dots, y_3(1+\kappa_3^l)) - C(t, y_1, \dots, y_3).$$

Solving for ζ^1 and ζ^2, we obtain

$$\begin{bmatrix} \lambda^1\zeta^1 \\ \lambda^2\zeta^2 \end{bmatrix} = \begin{bmatrix} \frac{\kappa_1^1-\kappa_2^1}{1+\kappa_1^1} & \frac{\kappa_1^2-\kappa_2^2}{1+\kappa_1^2} \\ \frac{\kappa_1^1-\kappa_3^1}{1+\kappa_1^1} & \frac{\kappa_1^2-\kappa_3^2}{1+\kappa_1^2} \end{bmatrix}^{-1} \begin{bmatrix} \mu_2 - \mu_1 + \sum_{l=1}^2 \lambda^l(\kappa_1^l - \kappa_2^l)\frac{\kappa_1^l}{1+\kappa_1^l} \\ \mu_3 - \mu_1 + \sum_{l=1}^2 \lambda^l(\kappa_1^l - \kappa_3^l)\frac{\kappa_1^l}{1+\kappa_1^l} \end{bmatrix},$$

provided that the matrix $\mathbf{A}_t$ is non-singular, i.e.,

$$a := |\mathbf{A}_t| = \frac{(\kappa_1^1-\kappa_2^1)(\kappa_1^2-\kappa_3^2) - (\kappa_1^1-\kappa_3^1)(\kappa_1^2-\kappa_2^2)}{(1+\kappa_1^1)(1+\kappa_1^2)} \neq 0.$$

Assuming that $a \neq 0$, we further obtain

$$\begin{aligned} \zeta^1 \;&=\; \frac{1}{a\lambda^1}\Big\{\frac{\kappa_1^2-\kappa_3^2}{1+\kappa_1^2}\Big(\mu_2 - \mu_1 + \sum_{l=1}^2 \lambda^l(\kappa_1^l-\kappa_2^l)\frac{\kappa_1^l}{1+\kappa_1^l}\Big) \\ &\quad - \frac{\kappa_1^2-\kappa_2^2}{1+\kappa_1^2}\Big(\mu_3 - \mu_1 + \sum_{l=1}^2 \lambda^l(\kappa_1^l-\kappa_3^l)\frac{\kappa_1^l}{1+\kappa_1^l}\Big)\Big\} \end{aligned}$$

and

$$\begin{aligned} \zeta^2 \;&=\; \frac{1}{a\lambda^2}\Big\{-\frac{\kappa_1^1-\kappa_2^1}{1+\kappa_1^1}\Big(\mu_2 - \mu_1 + \sum_{l=1}^2 \lambda^l(\kappa_1^l-\kappa_2^l)\frac{\kappa_1^l}{1+\kappa_1^l}\Big) \\ &\quad + \frac{\kappa_1^1-\kappa_3^1}{1+\kappa_1^1}\Big(\mu_3 - \mu_1 + \sum_{l=1}^2 \lambda^l(\kappa_1^l-\kappa_3^l)\frac{\kappa_1^l}{1+\kappa_1^l}\Big)\Big\}. \end{aligned}$$

We now have under $\mathbb{Q}$, for $i = 2, 3$,

$$dY_t^{i,1} = Y_{t-}^{i,1}\Big(\sum_{l=1}^2 \frac{\kappa_i^l - \kappa_1^l}{1+\kappa_1^l}\, d\widetilde{M}_t^l\Big)$$

and the vector $\mathbf{P}_t$ reduces to

$$\mathbf{P}_t = \begin{bmatrix} \frac{\Delta_1 C_t - \kappa_1^1 C_{t-}}{1+\kappa_1^1} & \frac{\Delta_2 C_t - \kappa_1^2 C_{t-}}{1+\kappa_1^2} \end{bmatrix}.$$

The replicating strategy for an FTDC can be found as in the preceding example.

4.4.3 Risk-free Asset, Default-free Asset, Defaultable Asset with Non-zero Recovery

We shall now consider an example that is equivalent to that used in Bielecki et al. [3]. Let Y^1 be a risk-free asset, Y^2 be a default-free asset and let Y^3 be a defaultable asset, so that

$$\begin{aligned} dY_t^1 &= rY_t^1\, dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\, dt + \sigma_2^1\, dW_t^1\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\, dt + \sigma_3^1\, dW_t^1 + \kappa_3^1\, dM_t^1\big). \end{aligned}$$

It is easily seen that the relative prices $Y^{2,1}$ and $Y^{3,1}$ satisfy

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1}\big((\mu_2 - r)\, dt + \sigma_2^1\, dW_t^1\big), \\ dY_t^{3,1} &= Y_{t-}^{3,1}\big((\mu_3 - r)\, dt + \sigma_3^1\, dW_t^1 + \kappa_3^1\, dM_t^1\big), \end{aligned}$$

and the pre-default pricing PDE of Proposition 4.1 reduces to

$$\partial_t C + \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i^1\sigma_j^1 y_i y_j \partial_{ij}C + ry_1\partial_1 C + \big(\mu_2 + \sigma_2^1\theta^1\big)y_2\partial_2 C$$
$$+ \big(\mu_3 + \sigma_3^1\theta^1 - \kappa_3^1\lambda^1(1+\zeta^1)\big)y_3\partial_3 C - rC + \lambda^1(1+\zeta^1)\,\Delta_1 C = 0$$

where

$$\Delta_1 C = Z_1(t, y_1, y_2, y_3(1+\kappa_3^1)) - C(t, y_1, y_2, y_3)$$

and θ^1 and ζ^1 are given by

$$\begin{bmatrix} \theta^1 \\ \lambda^1\zeta^1 \end{bmatrix} = \begin{bmatrix} -\sigma_2^1 & 0 \\ -\sigma_3^1 & -\kappa_3^1 \end{bmatrix}^{-1} \begin{bmatrix} \mu_2 - r \\ \mu_3 - r \end{bmatrix}.$$

Assuming that $\sigma_2^1 > 0$, $\kappa_3^1 \neq 0$ and $\lambda^1 > 0$, we obtain the unique solution for θ^1 and ζ^1

$$\theta^1 = \frac{r-\mu_2}{\sigma_2^1},$$
$$\zeta^1 = -\frac{\sigma_2^1(\mu_3 - r) - \sigma_3^1(\mu_2 - r)}{\sigma_2^1\kappa_3^1\lambda^1},$$

which coincides with the result found in Bielecki et al. [3]. Using the Girsanov theorem and recalling that

$$d\widetilde{W}_t^1 = dW_t^1 - \theta^1\,dt, \quad d\widetilde{M}_t^1 = dM_t^1 - \zeta^1\lambda^1\,dt,$$

we find that the dynamics of assets prices under $\mathbb{Q}$ are

$$dY_t^1 = rY_t^1\,dt,$$
$$dY_t^2 = Y_t^2(r\,dt + \sigma_2^1\,d\widetilde{W}_t^1),$$
$$dY_t^3 = Y_t^3(r\,dt + \sigma_3^1\,d\widetilde{W}_t^1 + \kappa_3^1\,d\widetilde{M}_t^1).$$

We can now substitute the values for θ^1 and ζ^1 into pre-default pricing PDE satisfied by the function $C(t, y_1, y_2, y_3)$ to obtain

$$\partial_t C + \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i^1\sigma_j^1 y_i y_j \partial_{ij}C + ry_1\partial_1 C + ry_2\partial_2 C + ry_3\partial_3 C - rC + \lambda^1(1+\zeta^1)\,\Delta_1 C = 0.$$

The replicating strategy satisfies

$$(\phi_t^2, \phi_t^3) = \frac{1}{Y_{t-}^1}\begin{bmatrix} \sigma_2^1 Y_{t-}^2\partial_2 C + \sigma_3^1 Y_{t-}^3\partial_3 C & \Delta_1 C_t \end{bmatrix} \begin{bmatrix} Y_{t-}^{1,2} & 0 \\ 0 & Y_{t-}^{1,3} \end{bmatrix} \begin{bmatrix} -\sigma_2^1 & 0 \\ -\sigma_3^1 & -\kappa_3^1 \end{bmatrix}^{-1}$$

so that

$$\phi_t^2 = \frac{\sigma_3^1 Y_t^2\Delta_1 C_t - (\sigma_2^1 Y_{t-}^2\partial_2 C + \sigma_3^1 Y_{t-}^3\partial_3 C)Y_{t-}^3}{\sigma_2^1\kappa_3^1 Y_{t-}^2 Y_{t-}^3}, \quad \phi_t^3 = \frac{\Delta_1 C_t}{Y_{t-}^3\kappa_3^1}.$$

As usual, the component ϕ_t^1 satisfies

$$\phi_t^1 = (Y_t^1)^{-1}\Big(C_t - \sum_{i=2}^{3}\phi_t^i Y_t^i\Big).$$

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