VOLATILITY STABILIZATION, DIVERSITY AND ARBITRAGE IN STOCHASTIC FINANCE

IOANNIS KARATZAS

Mathematics and Statistics Departments

Columbia University

`ik@math.columbia.edu`

Joint work with

Dr. E. Robert FERNHOLZ, C.I.O. of INTECH

Enhanced Investment Technologies, Princeton NJ

CONTENTS

REFERENCES

Fernholz, E.R. (2002). *Stochastic Portfolio Theory.* Springer-Verlag, New York.

Fernholz, E.R. & Karatzas, I. (2005). Relative arbitrage in volatility-stabilized markets. *Annals of Finance* **1**, 149-177.

Fernholz, E.R., Karatzas, I. & Kardaras, C. (2005). Diversity and arbitrage in equity markets. *Finance & Stochastics* **9**, 1-27.

Karatzas, I. & Kardaras, C. (2006). The numéraire portfolio and arbitrage in semimartingale markets. *Finance & Stochastics*, under revision.

Fernholz, E.R. & Karatzas, I. (2007) Stochastic Portfolio Theory: An Overview. *In preparation.* Preliminary form available upon request.

1. THE MARKET MODEL

Standard Model (Bachelier, Samuelson,...) for a Financial Market with n stocks and $d \geq n$ factors:

$$dX_i(t) = X_i(t)\left[b_i(t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)dW_\nu(t)\right],\ 1 \leq i \leq n.$$

Vector of rates-of-return: $b(\cdot) = (b_1(\cdot), \ldots, b_n(\cdot))'$.
Matrix of volatilities: $\sigma(\cdot) = (\sigma_{i\nu}(\cdot))_{1 \leq i \leq n,\, 1 \leq \nu \leq d}$.

Assumption: for every $T \in (0, \infty)$ we have

$$\sum_{i=1}^{n} \int_0^T \left(\Big| b_i(t) \Big| + \sum_{\nu=1}^{d} \Big(\sigma_{i\nu}(t)\Big)^2 \right) dt < \infty\,, \qquad \text{a.s.}$$

All processes are adapted to a given flow of information (or "filtration") $\mathbf{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$, which satisfies the usual conditions and may be strictly larger than the filtration generated by the driving Brownian motion $W(\cdot) = (W_1(\cdot), \ldots, W_d(\cdot))'$.

Solution of the equation

$$dX_i(t) = X_i(t)\left[b_i(t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)dW_\nu(t)\right]$$

for stock-price $X_i(\cdot)$ is written as

$$\underbrace{d\,(\log X_i(t)) \;\; = \;\; \gamma_i(t)\,dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\,dW_\nu(t)\,.}$$

Here $\mathsf{a}(\cdot) := \sigma(\cdot)\sigma'(\cdot)$ is the *covariance matrix*, and

$$\underbrace{\gamma_i(t) := b_i(t) - \frac{1}{2}\,\mathsf{a}_{ii}(t)}$$

the *growth-rate* of the i^{th} stock — in the sense

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)dt\right) = 0 \quad \text{a.s.},$$

at least when $\mathsf{a}(\cdot)$ is bounded.

2. PORTFOLIOS AND GROWTH RATES

Portfolio: A vector process $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))'$ which is adapted to $\mathbf{F}$ and fully-invested: no short-sales, no risk-free asset, to wit

$$\pi_i(t) \geq 0\,, \quad \sum_{i=1}^{n} \pi_i(t) = 1 \qquad \text{for all} \quad t \geq 0\,.$$

Value (wealth) $V^\pi(\cdot)$ of portfolio:

$$\frac{dV^\pi(t)}{V^\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dX_i(t)}{X_i(t))} = b^\pi(t)dt + \sum_{\nu=1}^{d} \sigma_\nu^\pi(t) dW_\nu(t).$$

Here

$$\underbrace{b^\pi(t) := \sum_{i=1}^{n} \pi_i(t) b_i(t)\,,} \qquad \underbrace{\sigma_\nu^\pi(t) := \sum_{i=1}^{n} \pi_i(t) \sigma_{i\nu}(t)\,,}$$

for $\nu = 1, \ldots, d$ are, respectively, the portfolio rate-of-return and the portfolio volatilities.

¶ The solution of this equation

$$\frac{dV^\pi(t)}{V^\pi(t)} = b^\pi(t)dt + \sum_{\nu=1}^{d} \sigma_\nu^\pi(t)dW_\nu(t)$$

is, very much like before:

$$\underbrace{d\,(\log V^\pi(t)) \;\; = \;\; \gamma^\pi(t)\,dt + \sum_{\nu=1}^{d} \sigma_\nu^\pi(t)\,dW_\nu(t)}\ .$$

• `Portfolio growth-rate` is

$$\underbrace{\gamma^\pi(t) := \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_*^\pi(t)}\ .$$

• `Excess growth-rate` is

$$\underbrace{\gamma_*^\pi(t) := \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\mathsf{a}_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\mathsf{a}_{ij}(t)\pi_j(t)\right)}\ .$$

This is a non-negative quantity, positive if $\pi_i(t) > 0$ for all $i = 1, \cdots, n$.

• `Portfolio variance` is

$$\mathsf{a}^{\pi\pi}(t) := \sum_{\nu=1}^{d} (\sigma_\nu^\pi(t))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\mathsf{a}_{ij}(t)\pi_j(t)\,.$$

• `Variance/Covariance Process`, relative to the portfolio $\pi(\cdot)$:

$$\mathfrak{A}_{ij}^\pi(t) \;:=\; \sum_{\nu=1}^{d} \Big(\sigma_{i\nu}(t) - \sigma_\nu^\pi(t)\Big)\Big(\sigma_{j\nu}(t) - \sigma_\nu^\pi(t)\Big)\;.$$

¶ We have the **invariance property**

$$\gamma_*^\pi(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\mathfrak{A}_{ii}^\rho(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\mathfrak{A}_{ij}^\rho(t)\pi_j(t)\right),$$

for *any* two portfolios $\pi(\cdot)$ and $\rho(\cdot)$, and its consequence:

$$\boxed{\gamma_*^\pi(t) \;=\; \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\,\mathfrak{A}_{ii}^\pi(t)}\;.$$

3. THE MARKET PORTFOLIO

Look at $X_i(t)$ as the capitalization of company i at time t (i.e., normalize always so that each company has exactly one share outstanding).

Then $X(t) := X_1(t) + \ldots + X_n(t)$ is the total capitalization of the entire market, and

$$\mu_i(t) := \frac{X_i(t)}{X(t)} = \frac{X_i(t)}{X_1(t) + \ldots + X_n(t)} > 0$$

the "relative capitalization" of the i^{th} company.

Clearly $\sum_{i=1}^n \mu_i(t) = 1$ for all $t \geq 0$, so $\mu(\cdot)$ is a portfolio process, called "`market portfolio`": ownership of $\mu(\cdot)$ is tantamount to ownership of the entire market, since $V^\mu(\cdot) \equiv \text{c}.X(\cdot)$; indeed,

$$\frac{dV^\mu(t)}{V^\mu(t)} = \sum_{i=1}^n \mu_i(t) \frac{dX_i(t)}{X_i(t)} = \sum_{i=1}^n \frac{dX_i(t)}{X(t)} = \frac{dX(t)}{X(t)}\,.$$

The excess growth rate of the market portfolio can then be interpreted as a measure of *intrinsic volatility* available in the market:

$$\overbrace{\underbrace{\gamma_*^\mu(t) \;=\; \frac{1}{2}\sum_{i=1}^{n} \mu_i(t)\,\mathfrak{A}_{ii}^\mu(t)}}\ ,$$

where

$$\mu_i(t) \;:=\; \frac{X_i(t)}{X(t)}\,, \qquad\qquad \sigma_\nu^\mu(t) := \sum_{i=1}^{n} \mu_i(t)\sigma_{i\nu}(t)$$

and

$$\mathfrak{A}_{ij}^\mu(t) \;:=\; \sum_{\nu=1}^{d} \Big(\sigma_{i\nu}(t) - \sigma_\nu^\mu(t)\Big)\Big(\sigma_{j\nu}(t) - \sigma_\nu^\mu(t)\Big)\ .$$

An average, according to capitalization weight, of the variances of individual stocks – not in absolute terms, but *relative to the market*.

This quantity will turn out to be very important in what follows.

4. RELATIVE ARBITRAGE

Given two portfolios $\pi(\cdot)$, $\rho(\cdot)$ and a real constant $T > 0$, we shall say that $\pi(\cdot)$ is an **arbitrage opportunity relative to $\rho(\cdot)$ over the time-horizon** $[0, T]$, if we have

$$\mathbb{P}\left[V^{\pi}(T) \geq V^{\rho}(T)\right] = 1\,,$$

$$\mathbb{P}\left[V^{\pi}(T) > V^{\rho}(T)\right] > 0$$

whenever the two portfolios start with the same initial fortune $V^{\pi}(0) = V^{\rho}(0) = 1$.

NOTE: With a "reasonable" (e.g. bounded) volatility structure, *the existence of relative arbitrage precludes the existence of an Equivalent Martingale Measure (EMM).*

- Indeed, if we can find a "market-price-of-risk" $\vartheta(\cdot)$ with

$$\sigma(\cdot)\,\vartheta(\cdot) = b(\cdot) \quad \text{and} \quad \int_0^T ||\vartheta(t)||^2\,dt < \infty \quad \text{a.s.,}$$

then it can be shown that the exponential process

$$Z(t) := \exp\left\{ -\int_0^t \vartheta'(s)\,dW(s) - \frac{1}{2}\int_0^t ||\vartheta(s)||^2\,ds \right\}$$

is a local (and super) martingale, but *not* a martingale: $\mathbb{E}[Z(T)] < 1$.

. Same for $Z(\cdot)X_i(\cdot)$: $\mathbb{E}[Z(T)X_i(T)] < X_i(0)$, $i = 1, \cdots, n$.

5. Functionally-generated portfolios

Start with a concave, smooth function $\mathbf{S} : \Delta^n_+ \to \mathbb{R}_+$, consider the portfolio $\pi(\cdot)$ generated by it:

$$\frac{\pi_i(t)}{\mu_i(t)} := D_i \log \mathbf{S}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) \cdot D_j \log \mathbf{S}(\mu(t)) \,.$$

Then an application of Itô's rule gives the "master equation"

$$\underbrace{\log\left(\frac{V^\pi(T)}{V^\mu(T)}\right) = \log\left(\frac{\mathbf{S}(\mu(T))}{\mathbf{S}(\mu(0))}\right) + \int_0^T \mathfrak{g}(t)\,dt}\,,$$

where, thanks to our assumptions,

$$\mathfrak{g}(t) := \frac{-1}{\mathbf{S}(\mu(t))} \sum_i \sum_j D^2_{ij}\mathbf{S}(\mu(t)) \cdot \mu_i(t)\mu_j(t)\, \mathfrak{A}^\mu_{ij}(t)$$

is a non-negative quantity.

Significance: Stochastic integrals have been excised, and we can begin to make comparisons that are valid with probability one (a.s.)...

6. SUFFICIENT INTRINSIC VOLATILITY LEADS TO ARBITRAGE RELATIVE TO THE MARKET

Proposition: Assume that over $[0,T]$ there is `"sufficient intrinsic volatility"` (excess growth):

$$\underbrace{\int_0^T \gamma_*^\mu(t)dt \geq \zeta T}, \quad \text{or} \quad \overbrace{\gamma_*^\mu(t) \geq \zeta\,, \quad 0 \leq t \leq T}$$

holds a.s., for some constant $\zeta > 0$. Take

$$T > T_* := \frac{\mathbf{H}(\mu(0))}{\zeta}, \quad \text{and} \quad \mathbf{H}(x) := -\sum_{i=1}^n x_i \log x_i$$

the Gibbs entropy function. Then the portfolio

$$\underbrace{\pi_i(t) := \frac{\mu_i(t)\,(c - \log \mu_i(t))}{\sum_{j=1}^n \mu_j(t)\,(c - \log \mu_j(t))}}, \quad i = 1, \cdots, n$$

is generated by the function $\mathbf{S}(x) := c + \mathbf{H}(x)$ on Δ_+^n ; and for $c > 0$ sufficiently large, it represents an arbitrage relative to the market.

¶ **`Sketch of Argument:`** Note $\mathbf{S}(x) := c + \mathbf{H}(x)$ is bounded both from above and below:

$$0 < c < \mathbf{S}(x) \leq c + \log n\,, \quad x \in \Delta_+^n\,.$$

The "master equation"

$$\underbrace{\log\left(\frac{V^\pi(T)}{V^\mu(T)}\right) = \log\left(\frac{\mathbf{S}(\mu(T))}{\mathbf{S}(\mu(0))}\right) + \int_0^T \mathfrak{g}(t)\,dt}$$

takes care of the rest, because now the integral of

$$\mathfrak{g}(\cdot) = \cdots = \frac{\gamma_*^\mu(\cdot)}{\mathbf{S}(\cdot)} \geq \frac{\gamma_*^\mu(\cdot)}{c + \log n}$$

dominates $\zeta T / (c + \log n)$: *if you have a constant wind on your back, sooner all later you'll overtake any obstacle* – e.g., the constant $\log\Big((c+\log n)/c\Big)$.

This leads to relative arbitrage for sufficiently large $T > 0$, indeed to $\mathbb{P}[V^\pi(T) > V^\rho(T)] = 1$.

. (Intuition: you can generate arbitrage if there is "enough volatility" in the market...)

Plot of cumulative excess growth $T \mapsto \int_0^T \gamma_*^\mu(t)\,dt$ over the period 1926-1999, in [FK] (2005).

7. NOTIONS OF MARKET DIVERSITY

MAJOR OPEN QUESTION: *Is such relative arbitrage possible over arbitrary time-horizons, under the conditions of this Proposition?*

Partial Answer #1: YES, if the variance/covariance matrix $\mathsf{a}(\cdot) = \sigma(\cdot)\sigma'(\cdot)$ has all its eigenvalues bounded away from zero and infinity: to wit, if we have (a.s.)

$$\kappa||\,\xi||^2 \le \xi'\mathsf{a}(t)\xi \le K||\,\xi||^2\,, \quad \forall\ t \ge 0,\ \xi \in \mathbb{R}^d \qquad (1)$$

for suitable constants $0 < \kappa < K < \infty$.

In this case one can show

$$\underbrace{\frac{\kappa}{2}\Big(1 - \mu_{(1)}(t)\Big) \le \gamma_*^\mu(t) \le 2K\Big(1 - \mu_{(1)}(t)\Big)}$$

for the maximal weight in the market

$$\mu_{(1)}(t) := \max_{1 \le i \le n} \mu_i(t)\,.$$

Thus, under the structural assumption of (1), the "sufficient intrinsic volatility" (a.s.) condition

$$\underbrace{\int_0^T \gamma_*^\mu(t)dt \geq \zeta T\,,} \quad \text{or} \quad \overbrace{\gamma_*^\mu(t) \geq \zeta\,, \quad 0 \leq t \leq T}$$

of the Proposition, is equivalent to the (a.s.) requirement of **market diversity**

$$\underbrace{\int_0^T \mu_{(1)}(t)dt \leq (1-\delta)T\,,} \quad \text{or} \quad \overbrace{\max_{0\leq t\leq T} \mu_{(1)}(t) \leq 1-\delta}$$

for some $\delta \in (0,1)$ (weak and strong, respectively).

¶ When (weak) diversity prevails, and with fixed $p \in (0,1)$, the simple *diversity-weighted* portfolio

$$\pi_i^{(p)}(t) := \frac{\Big(\mu_i(t)\Big)^p}{\sum_{j=1}^n \Big(\mu_j(t)\Big)^p}\,, \qquad \forall \quad i = 1, \ldots, n$$

also leads to arbitrage relative to the market, over sufficiently long time horizons.

. Appropriate modifications of this rule generate such arbitrage over *arbitrary* time-horizons.

8. AN ABSTRACT MODEL: STABILIZATION BY VOLATILITY

Partial Answer #2: YES, for the (non-diverse) `VOLATILITY-STABILIZED` model that we broach now.

Consider the abstract market model

$$\underbrace{d\Big(\log X_i(t)\Big) = \frac{\alpha\,dt}{2\mu_i(t)} + \frac{1}{\sqrt{\mu_i(t)}}\,dW_i(t)}$$

for $i = 1, \cdots, n$ with $d = n \geq 2$ and $\alpha \geq 0$. In other words, we assign the largest volatilities and the largest log-drifts to the smallest stocks . The model amounts to solving in the positive orthant of $\mathbb{R}^n$ the system of degenerate stochastic differential equations, for $i = 1, \cdots, n$:

$$dX_i(t) = \frac{1+\alpha}{2}\Big(X_1(t) + \cdots + X_n(t)\Big)\,dt$$
$$+ \sqrt{X_i(t)\Big(X_1(t) + \cdots + X_n(t)\Big)} \cdot dW_i(t)\,.$$

General theory: Bass & Perkins (TAMS 2002). Shows this system has a weak solution, unique in distribution, so the model is well-posed. **Better still: it is possible to describe this solution fairly explicitly, in terms of Bessel processes.**

✓ An elementary computation gives the quantities

$$\gamma_*^{\mu}(\cdot) \equiv \frac{n-1}{2} =: \zeta > 0\,, \qquad \mathsf{a}^{\mu\mu}(\cdot) \equiv 1$$

for the market portfolio $\mu(\cdot)$, and

$$\gamma^{\mu}(\cdot) \equiv \frac{(1+\alpha)n-1}{2} =: \gamma > 0\,.$$

Despite the erratic, widely fluctuating behavior of individual stocks, the overall market performance is remarkably stable. In particular, the total market capitalization is

$$X(t) = X_1(t) + \ldots + X_n(t) = x \cdot e^{\gamma t + B(t)}\,,$$

for the scalar Brownian motion

$$B(t) := \sum_{\nu=1}^{n} \int_0^t \sqrt{\mu_\nu(s)}\, dW_\nu(s)\,, \qquad 0 \le t < \infty\,.$$

¶ We call this phenomenon **stabilization by volatility**: big volatility swings for the smallest stocks, and smaller volatility swings for the largest stocks, end up stabilizing the overall market by producing constant, positive overall growth and variance.

(Note $\kappa = 1$ but $K = \infty$, so (1) fails.)

✓ The condition $\gamma_*^\mu(\cdot) \geq \zeta > 0$ of the Proposition on slide 12 is satisfied here, with $\zeta = (n-1)/2$. Thus the model admits arbitrage opportunities relative to the market, at least on time-horizons $[0, T]$ with $T > T_*$ with

$$T_* := \frac{2\,\mathbf{H}(\mu(0))}{n-1} < \frac{2\,\log n}{n-1}\,.$$

The upper estimate $(2 \log n)/(n-1)$ is a rather small number if $n = 5000$ as in Whilshire.

. This makes plausible the earlier claim, proved recently by A. Banner and D. Fernholz, that such arbitrage is now possible on *any* time-horizon.

• *What is the long-term-growth behavior of an* **individual** *stock?* A little bit of Stochastic Analysis provides the Representations

$$\underbrace{X_i(t) = \Big(\mathfrak{R}_i(\Lambda(t))\Big)^2, \qquad 0 \le t < \infty}, \quad i = 1, \cdots, n$$

and

$$\underbrace{X(t)} = X_1(t) + \cdots + X_n(t) = x\, e^{\gamma t + B(t)} = \underbrace{\Big(\mathfrak{R}(\Lambda(t))\Big)^2}.$$

Here

$$4\,\Lambda(t) := \int_0^t X(s)\, ds = x \int_0^t e^{\gamma s + B(s)}\, ds\,,$$

whereas $\mathfrak{R}_1(\cdot), \cdots, \mathfrak{R}_n(\cdot)$ are *independent* Bessel processes in `dimension` $m := 2(1+\alpha)$; that is,

$$d\mathfrak{R}_i(u) = \frac{m-1}{2\mathfrak{R}_i(u)}\, du + d\widehat{W}_i(u)$$

with $\widehat{W}_1(\cdot), \cdots, \widehat{W}_n(\cdot)$ independent scalar Brownian motions. Finally,

$$\mathfrak{R}(u) := \sqrt{\Big(\mathfrak{R}_1(u)\Big)^2 + \cdots + \Big(\mathfrak{R}_n(u)\Big)^2}$$

is Bessel process in `dimension` mn.

We are led to the *skew representation* (I. Goia)

$$\boxed{\mathfrak{R}_i^2(u) \,=\, \mathfrak{R}^2(u) \cdot \mu_i \left(4 \int_0^u \frac{du}{\mathfrak{R}^2(u)} \right), \quad 0 \le u < \infty}$$

where the vector $\mu(\cdot) = (\mu_i(\cdot))_{i=1}^n$ of market-weights $\mu_i(\cdot) = (\mathfrak{R}_i^2/\mathfrak{R}^2)(\Lambda(\cdot))$ is *independent* of the Bessel process $\mathfrak{R}(\cdot)$.

This $\mu(\cdot)$ is shown to be a vector <u>`Jacobi process`</u> with values in Δ_+^n and dynamics

$$d\mu_i(t) = (1+\alpha)\Big(1-n\mu_i(t)\Big)dt + \Big(1-\mu_i(t)\Big)\sqrt{\mu_i(t)}\, d\beta_i(t)$$

$$- \mu_i(t) \sum_{j \ne i} \sqrt{\mu_j(t)}\, d\beta_j(t)\,, \quad i = 1, \cdots, n$$

(variances $\mu_i(1-\mu_i)$, covariances $-\mu_i\mu_j$).

This also suggests the distribution of the vector

$$\left(\frac{Q_1}{Q_1 + \cdots + Q_n}, \cdots, \frac{Q_n}{Q_1 + \cdots + Q_n} \right),$$

where $Q_1, \cdots, Q_n$ are independent random variables with common distribution

$$\frac{2^{-(1+\alpha)}}{\Gamma(1+\alpha)} \, q^{\alpha} e^{-q/2} \, dq \,, \qquad 0 < q < \infty \,,$$

(chi-square with "$2(1+\alpha)$-degrees-of-freedom"), as the invariant measure for the Δ^n_+-valued diffusion $\mu(\cdot) = (\mu_i(\cdot))_{i=1}^n$.

¶ From these representations, one obtains the (a.s.) *long-term growth rates of the entire market and of the largest stock*

$$\underline{\lim_{T\to\infty} \frac{1}{T} \log X(T) = \lim_{T\to\infty} \frac{1}{T} \log \left(\max_{1\le i\le n} X_i(T) \right) = \gamma \,;}$$

the *long-term growth rates for individual stocks*

$$\underbrace{\lim_{T\to\infty} \frac{1}{T} \log X_i(T) = \gamma}\,, \quad i = 1, \cdots, n \qquad (2)$$

for $\alpha > 0$; their long-term volatilities

$$\underbrace{\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{dt}{\mu_i(t)} = \frac{2\gamma}{\alpha} = n + \frac{n-1}{\alpha}}$$

(for $\alpha > 0$, using the Birkhoff ergodic theorem); that this model is not diverse; and much more...

NOTE: When $\alpha = 0$, the equation (2) holds only `in probability`; the (a.s.) limit-superior is (γ), whereas the (a.s.) limit-inferior is $(-\infty)$. (Spitzer's 0-1 law for planar Brownian motion). Crashes.... Failure of diversity... .

8. SOME CONCLUDING REMARKS

We have exhibited simple conditions, such as "sufficient level of intrinsic volatility" and "diversity", which lead to arbitrages relative to the market.

These conditions are **descriptive** as opposed to normative, and can be tested from the predictable characteristics of the model posited for the market. In contrast, familiar assumptions, such as the existence of an equivalent martingale measure (EMM), are **normative** in nature, and *cannot* be decided on the basis of predictable characteristics in the model; see example in [KK] (2006).

The existence of such relative arbitrage is not the end of the world. Under reasonably general conditions, one can still work with appropriate "deflators" for the purposes of hedging derivatives and of portfolio optimization.

Considerable computational tractability is lost, as the marvelous tool that is the EMM goes out of the window; nevertheless, big swaths of the field of Mathematical Finance remain totally or mostly intact, and completely new areas and issues thrust themselves onto the scene.