Law of the Minimal Price

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Platen, E.: A benchmark approach to asset management. *J. Asset Managem.* **6**(6), 390-405 (2006).

Law of One Price

“All replicating portfolios of a payoff have the same price!”

Debreu (1959), Sharpe (1964), Lintner (1965),

Merton (1973a, 1973b), Ross (1976), Harrison & Kreps (1979),

Cochrane (2001), ...

will be, in general, violated under the benchmark approach.

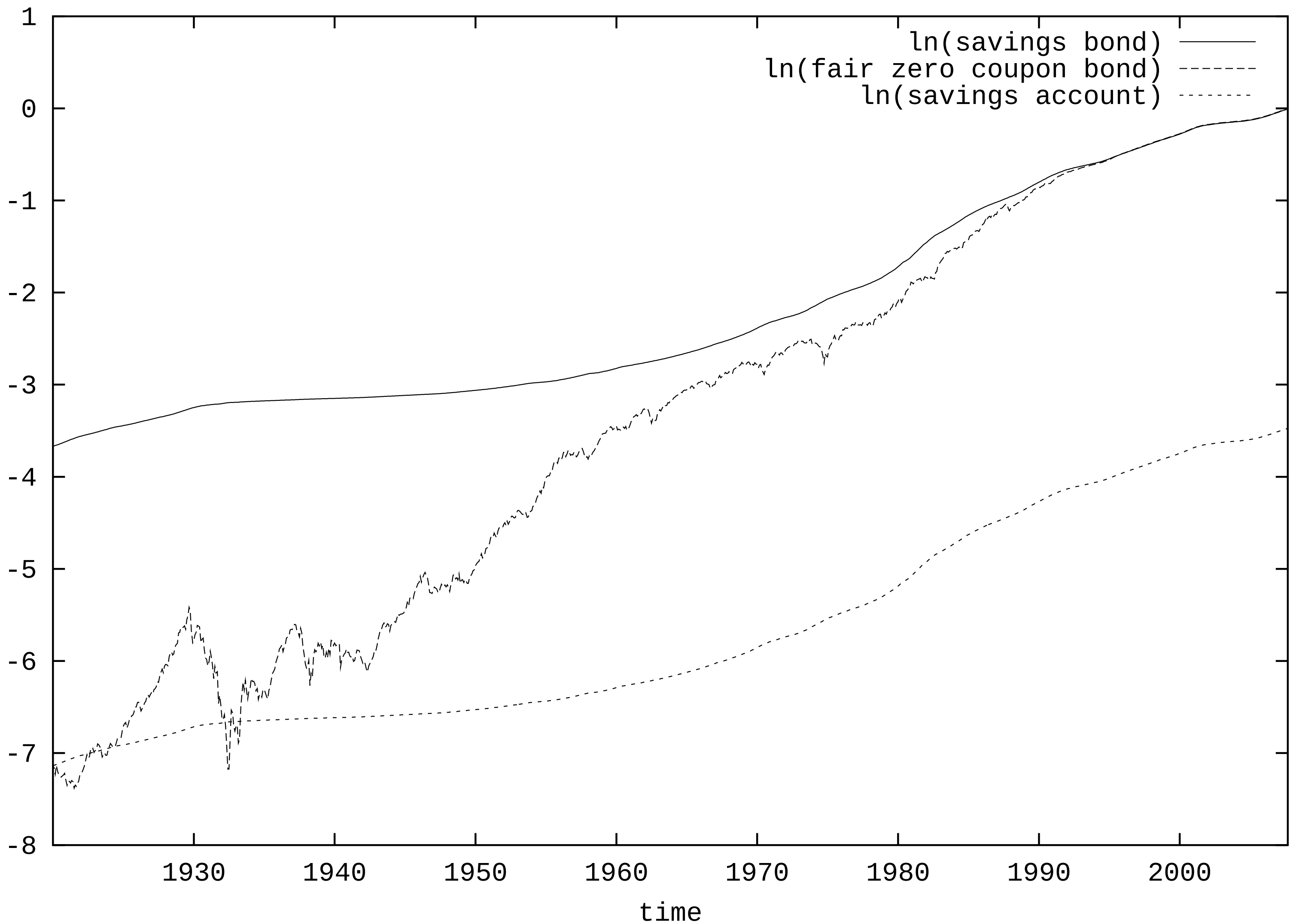

Figure 1: Logarithms of savings bond, fair zero coupon bond and savings account.

Two Asset Market   Example

- trading times $t_i = i\,h,\ h > 0$

- **risky asset** S^1 (S&P500 accumulation index)

  asset ratio $A^1_{t_i,h} = \frac{S^1_{t_i+h}}{S^1_{t_i}} \in (0,\infty)$

  can reach **any** finite strictly positive value

- **savings account** S^0 (US savings account)

  asset ratio $A^0_{t_i,h} = \frac{S^0_{t_i+h}}{S^0_{t_i}} > 0$

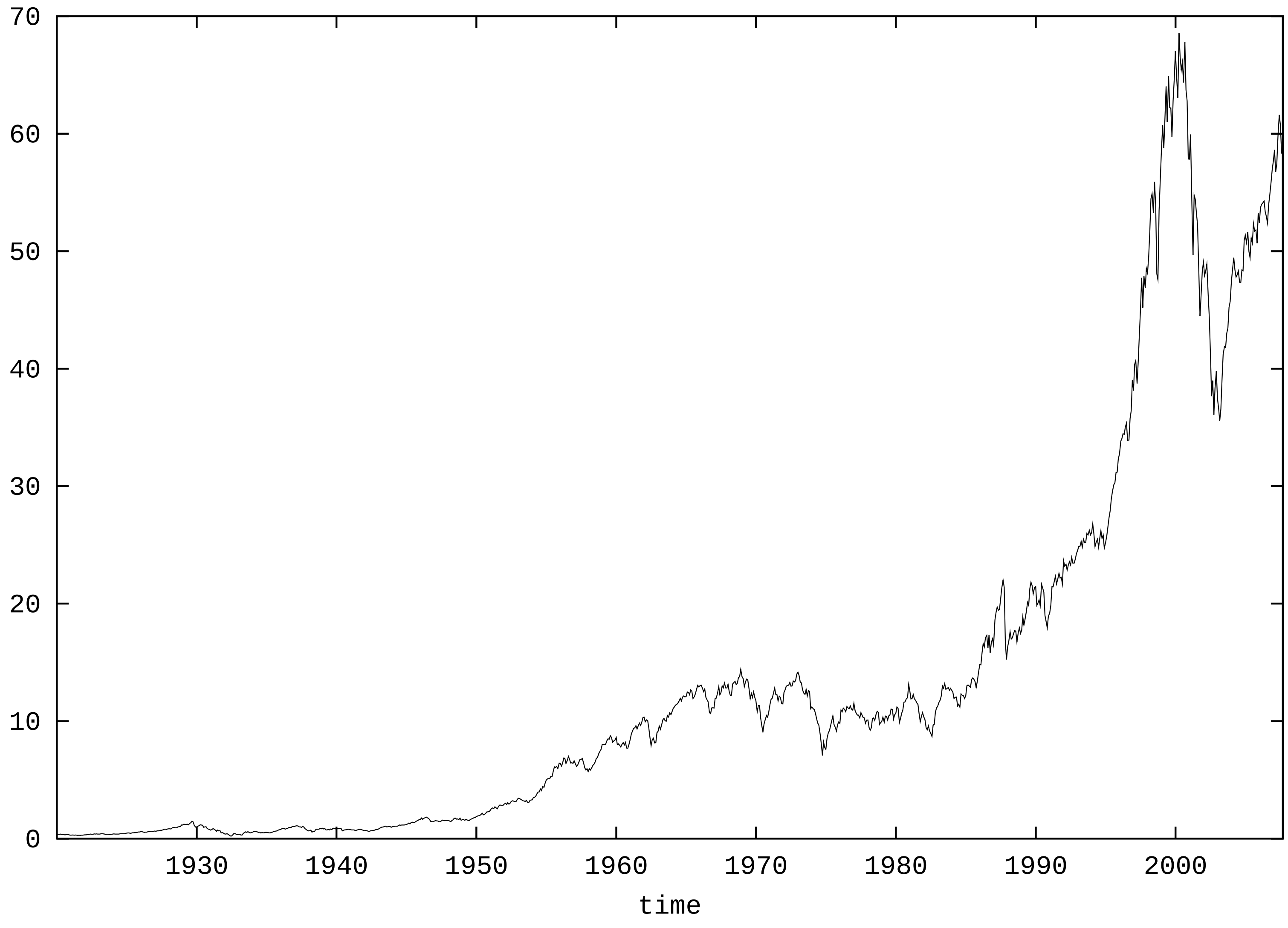

Figure 2: Discounted S&P500.

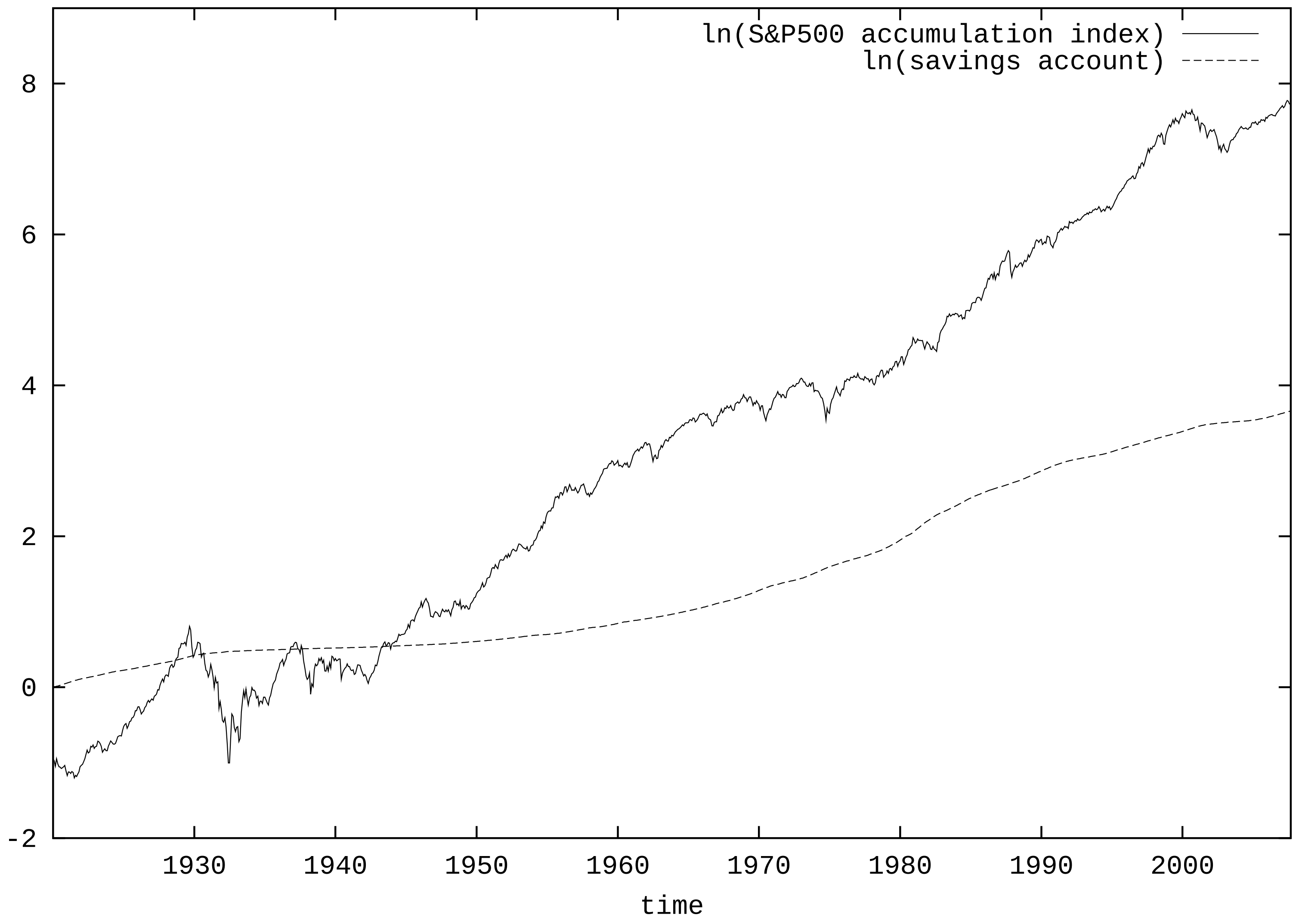

Figure 3: ln(S&P500 accumulation index) and ln(savings account).

- **portfolio S^δ**

strict positivity of S^δ $\quad \Longleftrightarrow \quad \pi^0_{\delta,t_i} = \delta^0_{t_i} \frac{S^0_{t_i}}{S^\delta_{t_i}} \in [0,1]$

$$A^\delta_{t_i,h} = \frac{S^\delta_{t_i+h}}{S^\delta_{t_i}} = \pi^0_{\delta,t_i} A^0_{t_i,h} + \left(1 - \pi^0_{\delta,t_i}\right) A^1_{t_i,h}$$

- find **best performing** strictly positive portfolio

- **expected growth**

$$g^{\delta}_{t_i,h} \;=\; E_{t_i}\left(\ln\left(A^{\delta}_{t_i,h}\right)\right)$$

$$\frac{\partial g^{\delta}_{t_i,h}}{\partial \pi^0_{\delta,t_i}} = E_{t_i}\left(\frac{A^0_{t_i,h} - A^1_{t_i,h}}{A^{\delta}_{t_i,h}}\right)$$

second derivative negative

$\Longrightarrow$ one genuine maximum at $\pi^0_{\underline{\delta},t_i} \in (-\infty,\infty)$ for

$$\frac{\partial g^{\delta}_{t_i,h}}{\partial \pi^0_{\delta,t_i}} = 0$$

- **Three cases:**

(i) $\pi^{0}_{\underline{\delta},t_i} \in [0,1]$ classical case

(ii) $\pi^{0}_{\underline{\delta},t_i} < 0$ savings account performs poorly, risk premium

$\Longrightarrow$ **index** is best performing portfolio

(iii) $\pi^{0}_{\underline{\delta},t_i} > 1$ index performs poorly

$\Longrightarrow$ **savings account** is best performing portfolio

- **classical case (i):** $\pi^{0}_{\underline{\delta},t_i} \in [0,1]$

$\Longrightarrow$

$$E_{t_i}\left(\frac{A^{1}_{t_i,h}}{A^{\underline{\delta}}_{t_i,h}}\right) = E_{t_i}\left(\frac{A^{0}_{t_i,h}}{A^{\underline{\delta}}_{t_i,h}}\right) = 1$$

$\Longrightarrow$ genuine maximum $g^{\delta_*}_{t_i,h} = g^{\underline{\delta}}_{t_i,h}$

$S^{0}_{t_i}$ and $S^{1}_{t_i}$ are constituents of $S^{\delta_*}_{t_i}$

$\frac{S^{0}_{t_i}}{S^{\delta_*}_{t_i}}$ and $\frac{S^{1}_{t_i}}{S^{\delta_*}_{t_i}}$ are martingales $\widehat{=}$ fair

$\Longrightarrow$ all benchmarked portfolios are fair

$\Longrightarrow$ Law of One Price holds

- **other cases**

rewrite first order condition

$$\frac{\partial g^{\delta}_{t_i,h}}{\partial \pi^0_{\delta,t_i}} = E_{t_i}\left(Q_{t_i,h} - \pi^0_{\delta,t_i} \, \frac{(Q_{t_i,h})^2}{1 + \pi^0_{\delta,t_i} \, Q_{t_i,h}} \right) = 0$$

with

$$Q_{t,h} = \frac{\frac{S^0_{t+h}}{S^1_{t+h}}}{\frac{S^0_t}{S^1_t}} - 1$$

$\Longrightarrow$

$$\pi^0_{\underline{\delta},t} = \frac{\lim_{h\to 0} \frac{1}{h} \, E_t(Q_{t,h})}{\lim_{h\to 0} \frac{1}{h} \, E_t\left(\frac{(Q_{t,h})^2}{1+\pi^0_{\underline{\delta},t} \, Q_{t,h}} \right)}$$

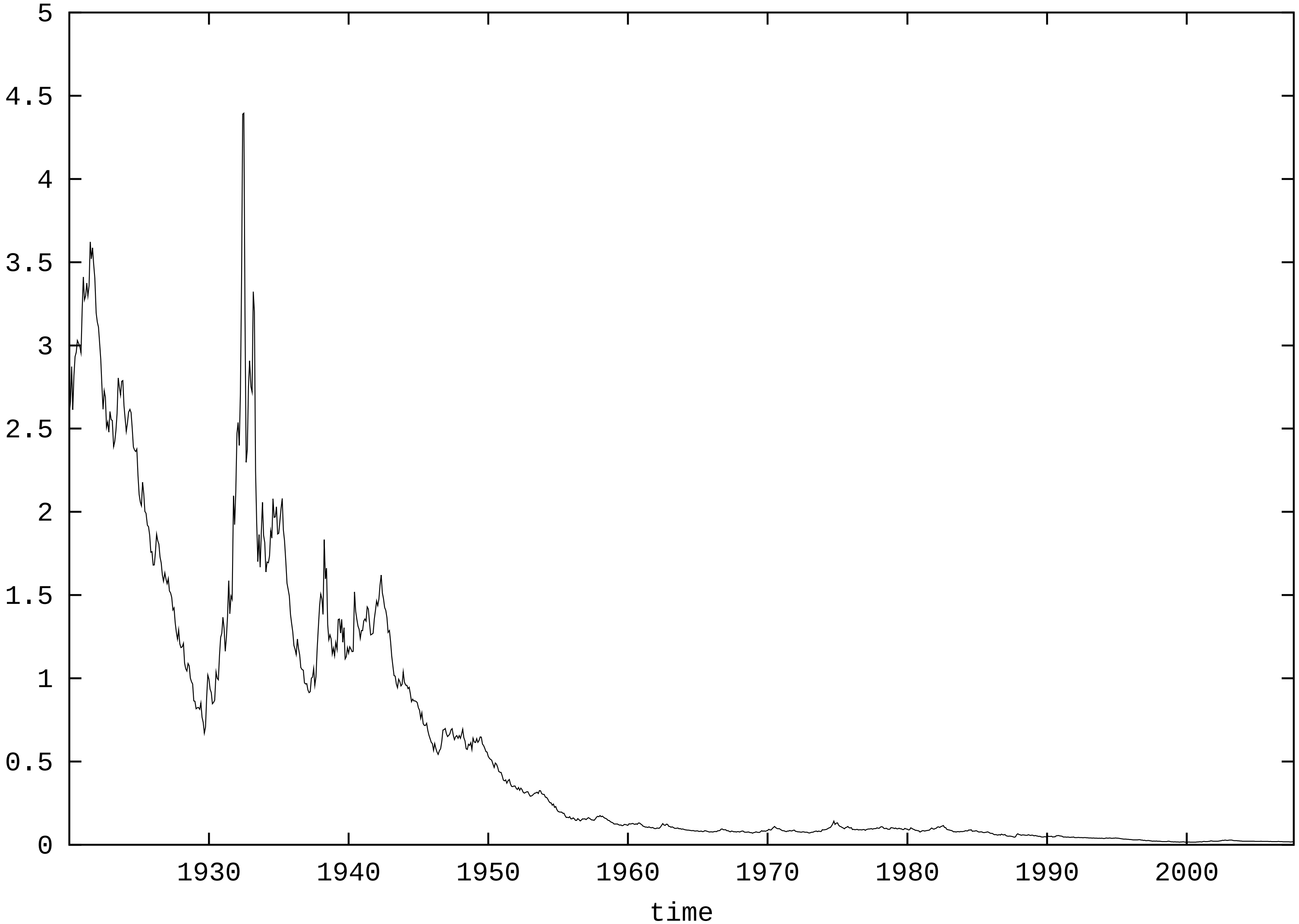

Figure 4: US-benchmarked savings account.

- annualized returns of benchmarked savings account

  $n = 1052$

$$\hat{\mu} = \frac{1}{n+1} \sum_{i=0}^{n} \frac{1}{h} Q_{t_i,h} \approx -0.0396$$

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=0}^{n} \frac{1}{h} (Q_{t_i,h})^2} \approx 0.19$$

negative fraction $\pi^0_{\underline{\delta},t} = \frac{\hat{\mu}}{\hat{\sigma}^2} \approx -1.1$

$\Longrightarrow$ savings account unlikely to be fair

Extreme Maturity Bond

- need model with downward trending $\frac{S_t^0}{S_t^{\delta_*}}$

  to reflect reality

- assume short rate deterministic

- **savings bond**

$$P^*(t,T) = \frac{S_t^0}{S_T^0}$$

- select index S^1 as numeraire portfolio S^{δ_*}

- benchmarked fair zero coupon bond

$$\hat{P}(t,T) = \frac{P(t,T)}{S_t^{\delta_*}} = E_t\left(\frac{1}{S_T^{\delta_*}}\right)$$

martingale

$\Longrightarrow$ real world pricing formula

$$P(t,T) = S_t^{\delta_*}\, E_t\left(\frac{1}{S_T^{\delta_*}}\right)$$

- **discounted numeraire portfolio**

$$\bar{S}_t^{\delta_*} = \frac{S_t^{\delta_*}}{S_t^0}$$

for continuous market generally satisfies SDE

$$d\bar{S}_t^{\delta_*} = \alpha_t\, dt + \sqrt{\bar{S}_t^{\delta_*}\, \alpha_t}\, dW_t$$

is time transformed squared Bessel process

- model the drift of discounted index as

$$\alpha_t = \frac{\alpha}{\eta} \exp\{\eta\, t\}$$

$\Longrightarrow$ **Minimal Market Model**

MMM, see Pl. & Heath (2006)

- **net growth rate**

$\eta \approx 0.0511$ with R^2 of 0.88

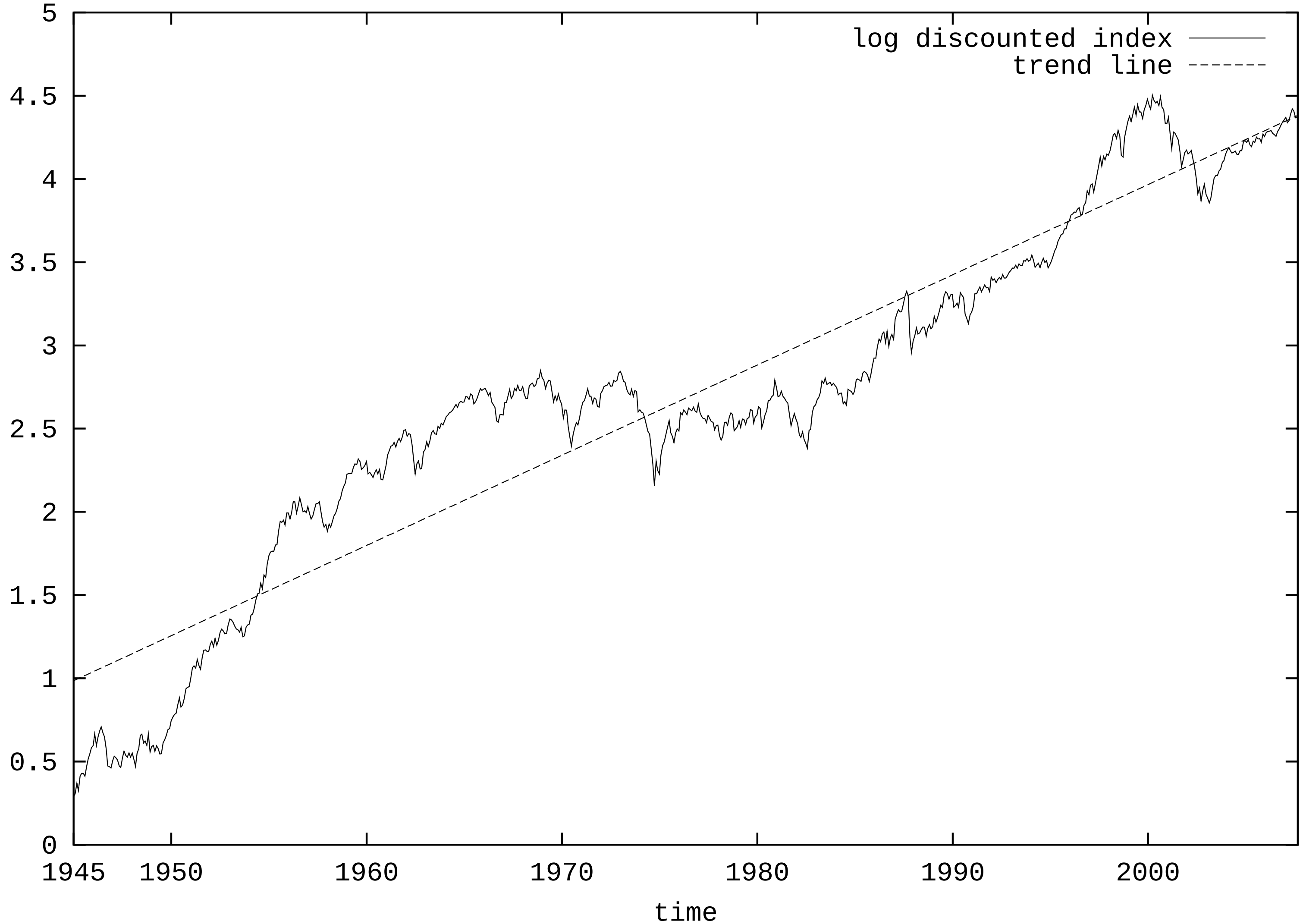

Figure 5: Logarithm of discounted index.

- **normalized index**

$$Y_t = \frac{\bar{S}_t^{\delta_*}}{\alpha_t}$$

$$dY_t = (1 - \eta\, Y_t)\, dt + \sqrt{Y_t}\, dW_t$$

- **quadratic variation of $\sqrt{Y_t}$**

$$d\sqrt{Y_t} = \ldots + \frac{1}{2}\, dW_t$$

$$V_{t,h} = \sum_{\ell=1}^{i_t} \left(\sqrt{Y_{t_\ell}} - \sqrt{Y_{t_{\ell-1}}} \right)^2 \approx \left[\sqrt{Y} \right]_t = \frac{t}{4}$$

- **scaling parameter** $\alpha \approx 0.01429$ with R^2 of 0.995

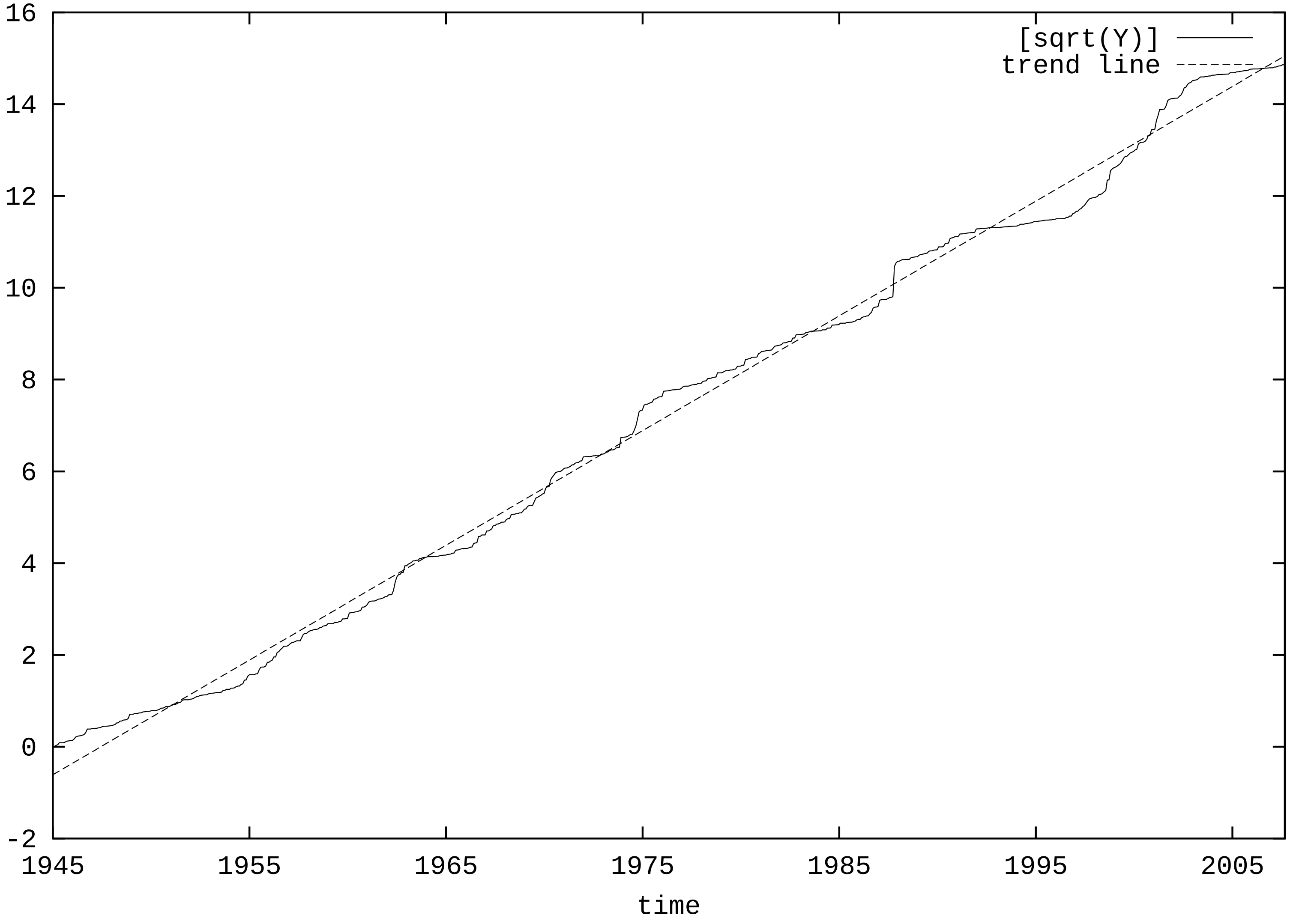

Figure 6: Quadratic Variation of $\sqrt{Y_t}$.

- **fair zero coupon bond (MMM)**

$$P(t,T) = P^*(t,T)\left(1 - \exp\left\{-\frac{2\,\eta\,\bar{S}_t^{\delta_*}}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}\right\}\right)$$

- **initial prices** in 1920:

$$P^*(0,T) = 0.025496$$

$$P(0,T) = 0.000795$$

$$\frac{P(0,T)}{P^*(0,T)} < 0.0312 \qquad \Longrightarrow \qquad 3.12\%$$

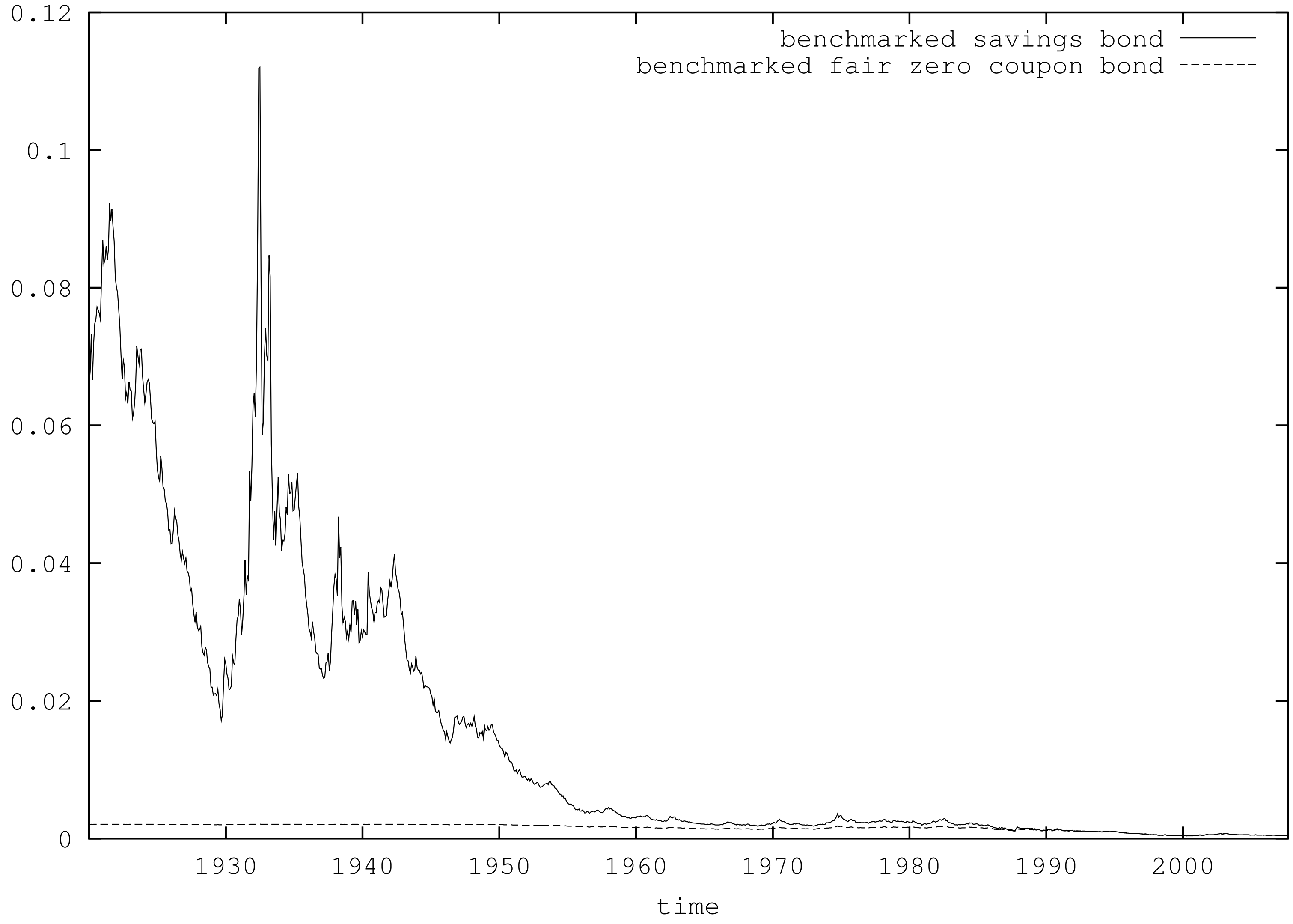

Figure 7: Benchmarked savings bond and benchmarked fair zero coupon bond.

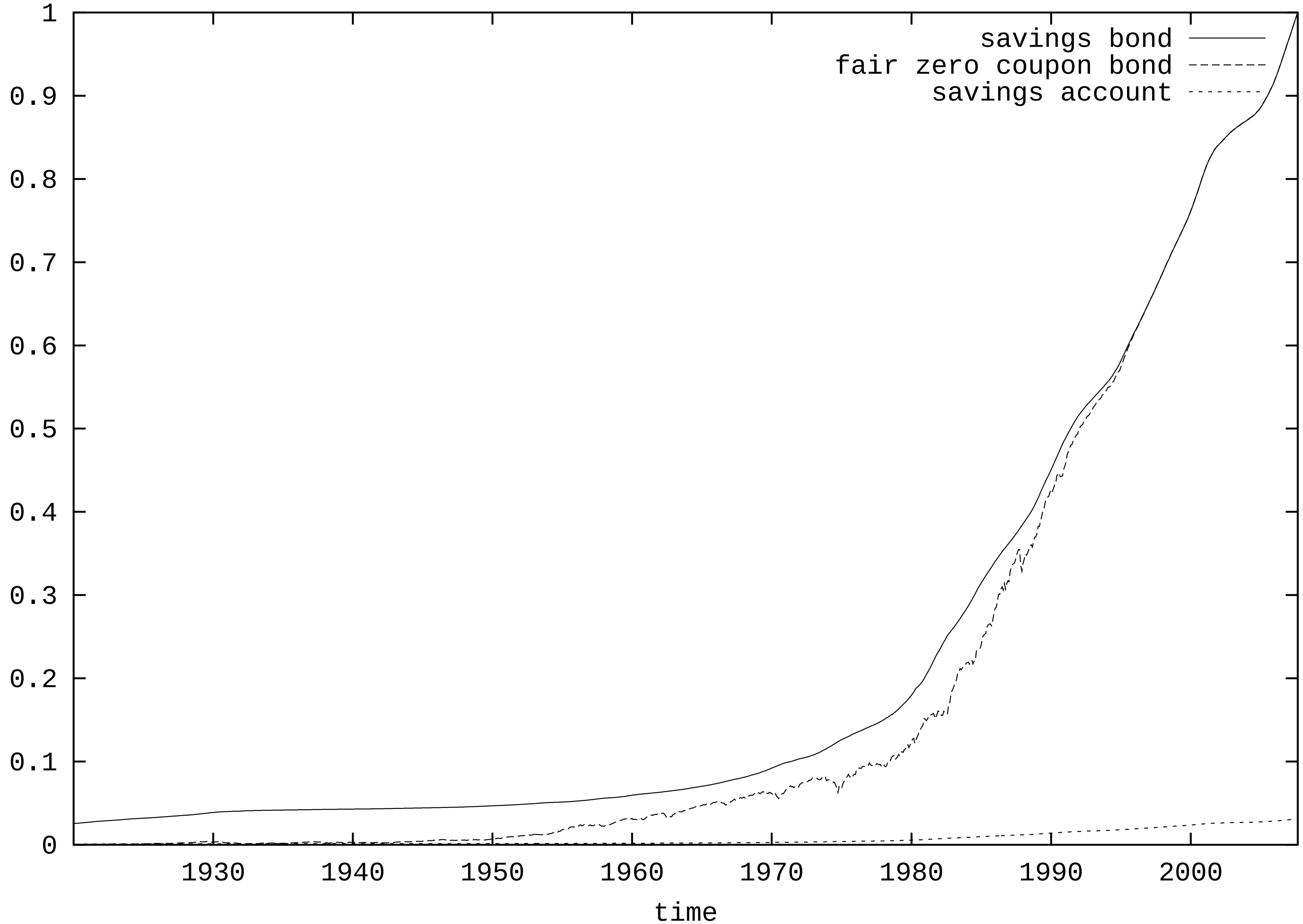

Figure 8: Savings bond, fair zero coupon bond and savings account.

- **self-financing hedge portfolio** hedge ratio

$$\delta^*_t = \frac{\partial P(t,T)}{\partial \bar{S}^{\delta_*}_t}$$

$$= P^*(t,T)\, \exp\left\{\frac{-2\,\eta\,\bar{S}^{\delta_*}_t}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}\right\}$$

$$\times \frac{2\,\eta}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}$$

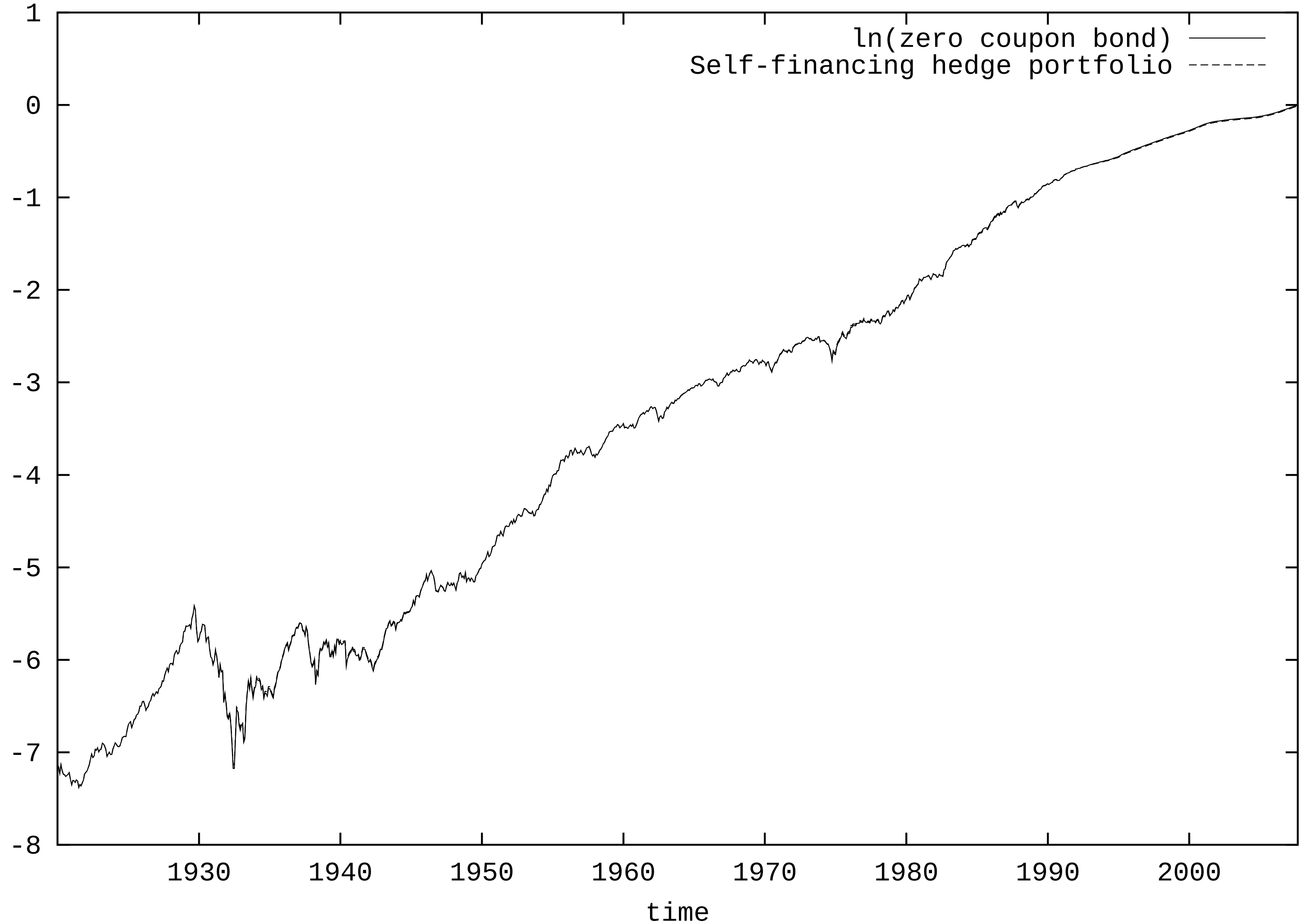

Figure 9: Logarithm of zero coupon bond and self-financing hedge portfolio.

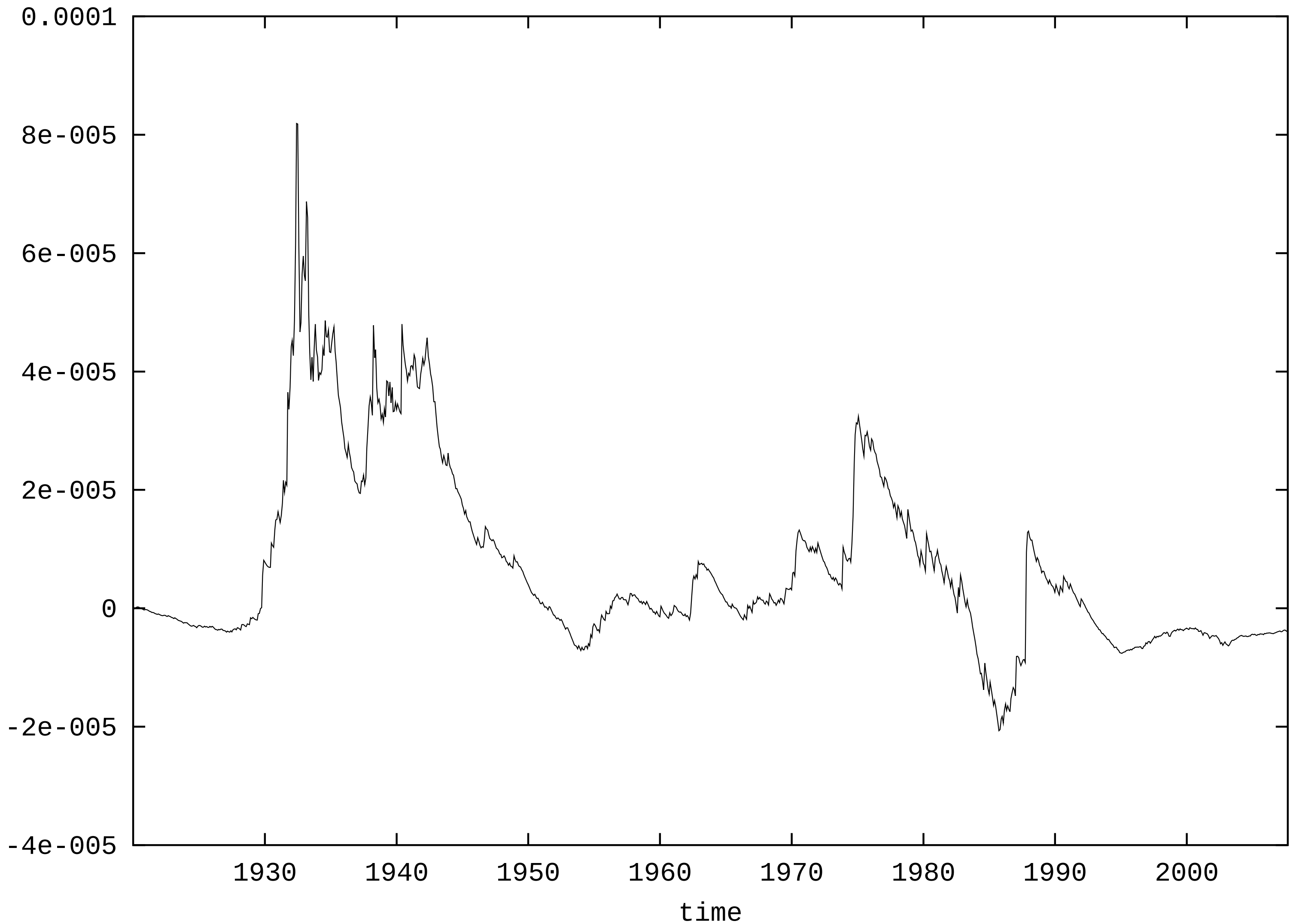

Figure 10: Benchmarked P&L.

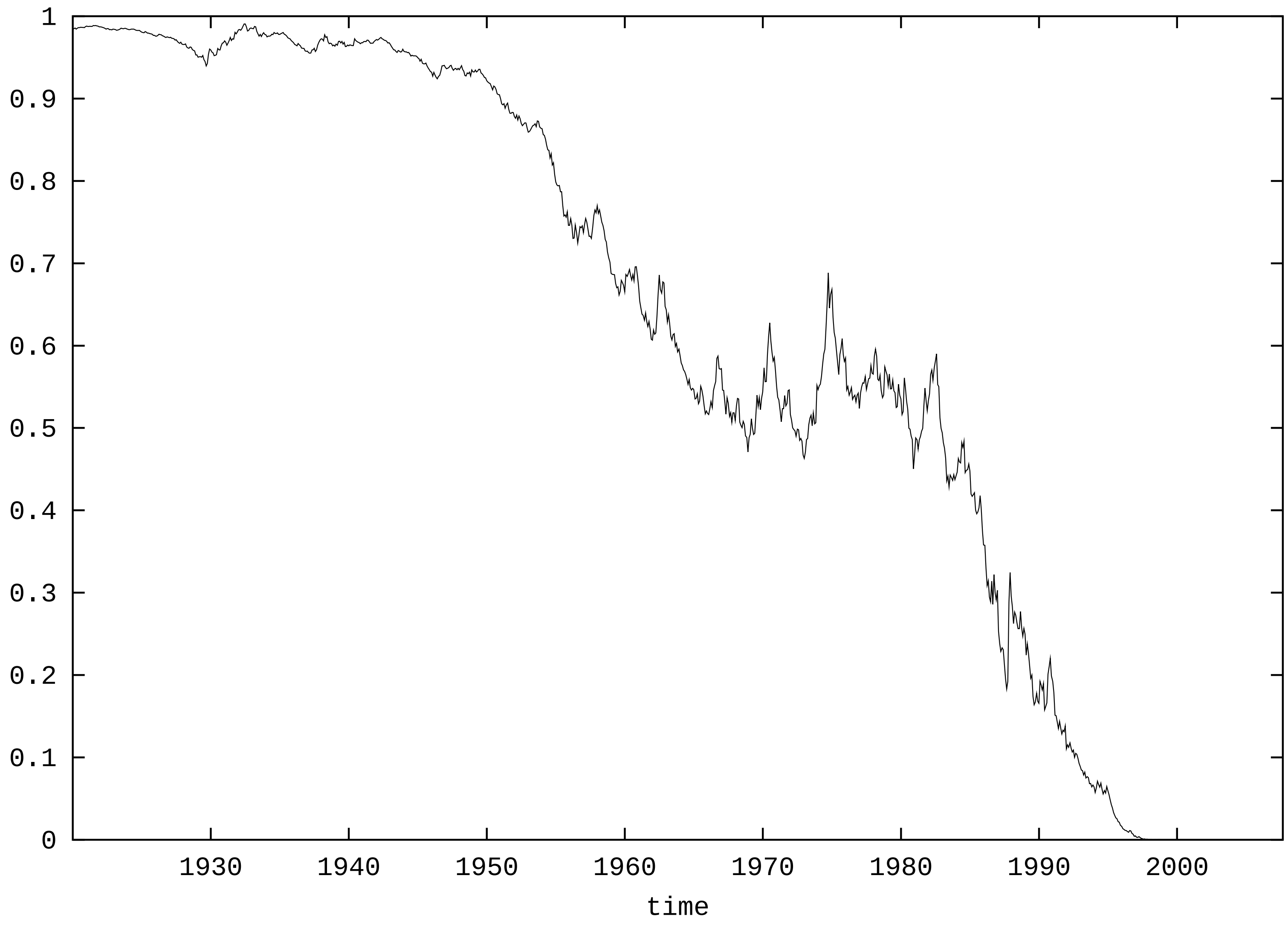

Figure 11: Fraction invested in the index.

Financial Market

- jth primary security account

  S_t^j

  $j \in \{0, 1, \ldots, d\}$

- savings account

  S_t^0

  $t \geq 0$

- **strategy**

$$\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \ldots, \delta_t^d)^\top,\ t \geq 0\}$$

predictable

- **portfolio**

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j\, S_t^j$$

- **self-financing**

$$dS_t^\delta = \sum_{j=0}^{d} \delta_t^j\, dS_t^j$$

Numeraire Portfolio

Definition 1 $S^{\delta_*} \in \mathcal{V}_x^+$ numeraire portfolio *if*

$$E_t\left(\frac{\frac{S^{\delta}_{t+h}}{S^{\delta_*}_{t+h}}}{\frac{S^{\delta}_{t}}{S^{\delta_*}_{t}}} - 1\right) \leq 0$$

for all nonnegative S^{δ} *and* $t, h \in [0, \infty)$.

- S^{δ_*} "best" performing portfolio

Long (1990), Becherer (2001), Pl. (2002, 2006),

Bühlmann & Pl. (2003), Goll & Kallsen (2003), Karatzas & Kardaras (2007)

Main Assumption of the Benchmark Approach

Assumption 2 *There* ***exists*** *a numeraire portfolio* $S^{\delta_*} \in \mathcal{V}_x^+$.

Supermartingale Property

- **benchmarked value**

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}$$

Corollary 3 *For nonnegative* S^δ

$$\hat{S}_t^\delta \geq E_t\left(\hat{S}_s^\delta\right)$$

$0 \leq t \leq s < \infty$

nonnegative $\hat{S}^\delta$ *supermartingale*

Definition 4 *Price is* fair *if, when benchmarked, forms martingale.*

$$\hat{S}_t^\delta = E_t\left(\hat{S}_s^\delta\right)$$

for $0 \leq t \leq s < \infty$.

Lemma 5 *The* ***minimal nonnegative supermartingale*** *that reaches a given benchmarked contingent claim* ***is a martingale****.*

see Pl. & Heath (2006)

Law of the Minimal Price

Theorem 6 *If a fair portfolio replicates a nonnegative payoff, then this represents the minimal replicating portfolio.*

- least expensive
- minimal hedge
- economically correct price in a competitive market

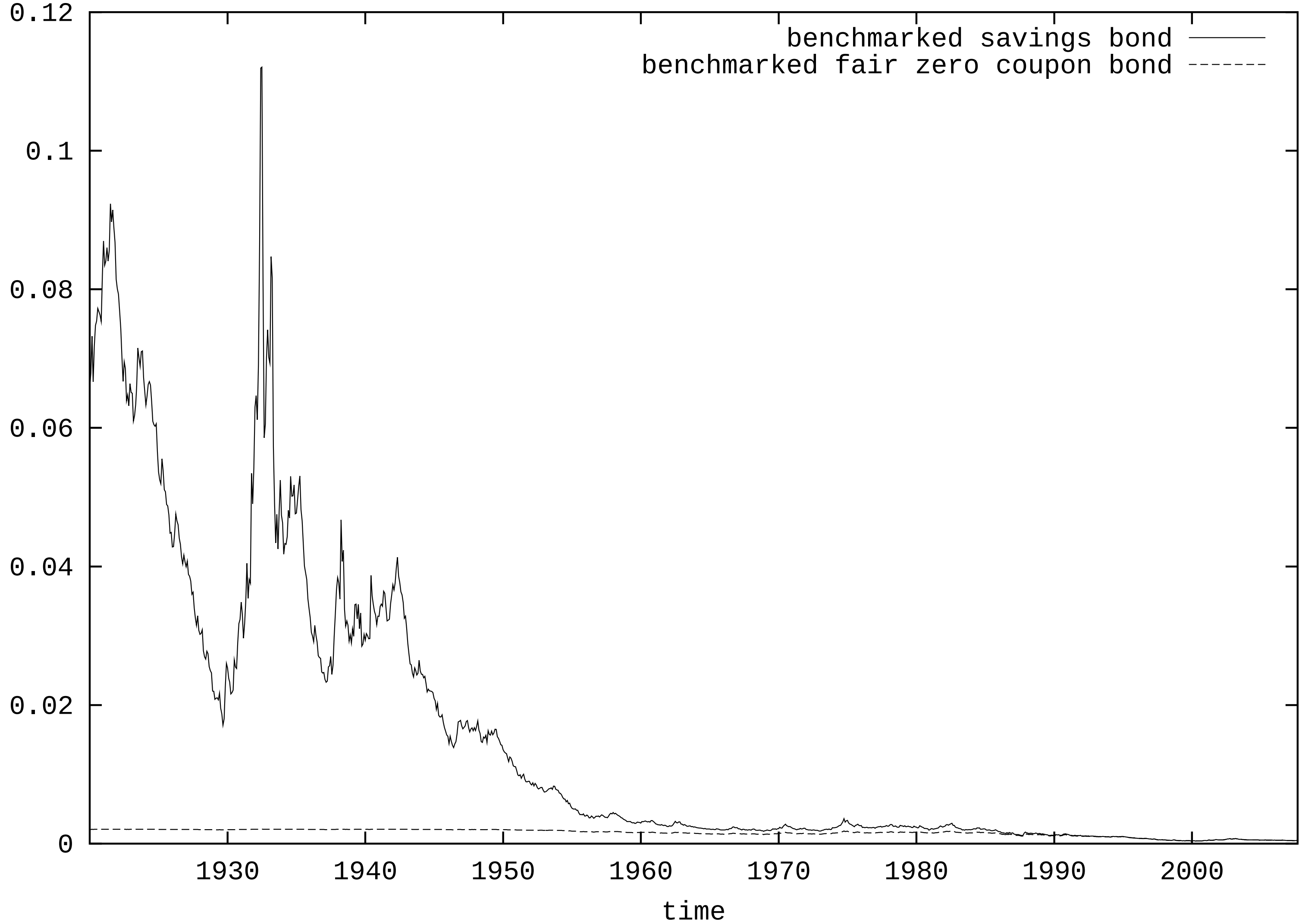

Figure 12: Benchmarked savings bond and benchmarked fair zero coupon bond.

- **claim**

  H_T

$$E_0\left(\frac{H_T}{S_T^{\delta_*}}\right) < \infty$$

Corollary 7

Minimal price for replicable H_T *is given by*

real world pricing formula

$$S_t^{\delta_H} = S_t^{\delta_*}\, E_t\left(\frac{H_T}{S_T^{\delta_*}}\right).$$

- **normalized benchmarked savings account**

$$\Lambda_T = \frac{\hat{S}^0_T}{\hat{S}^0_0}$$

$$1 = \Lambda_0 \geq E_0(\Lambda_T)$$

- **real world pricing formula** $\Longrightarrow$

$$S_0^{\delta_H} = E_0\left(\Lambda_T \, \frac{S^0_0}{S^0_T} \, H_T\right)$$

$\Longrightarrow$

$$S_0^{\delta_H} \leq \frac{E_0\left(\Lambda_T \, \frac{S^0_0}{S^0_T} \, H_T\right)}{E_0(\Lambda_T)}$$

similar for any numeraire

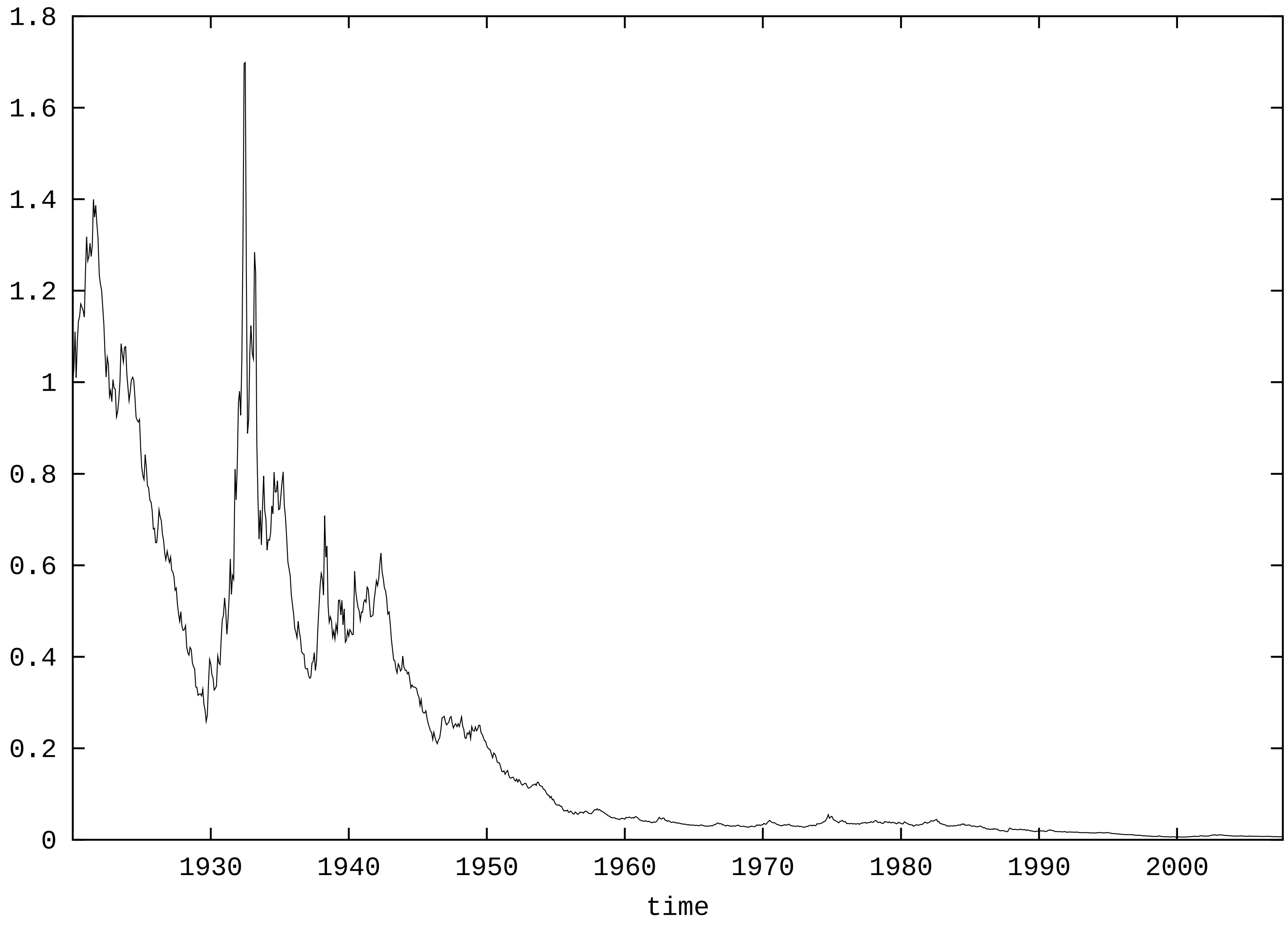

Figure 13: Candidate Radon-Nikodym derivative of hypothetical risk neutral measure of real market.

- **special case when savings account is fair:**

$$\Longrightarrow \qquad \Lambda_T = \frac{dQ}{dP} \text{ forms martingale;} \quad E_0(\Lambda_T) = 1;$$

equivalent risk neutral probability measure Q exists;

Bayes' formula $\quad \Longrightarrow$
risk neutral pricing formula

$$S_0^{\delta_H} = E_0^Q\left(\frac{S_0^0}{S_T^0}\, H_T\right)$$

Harrison & Kreps (1979), Ingersoll (1987),

Constatinides (1992), Duffie (2001), Cochrane (2001), $\ldots$

- otherwise "risk neutral price" $\geq$ real world price

- **long term growth rate**

$$\boldsymbol{g^{\delta} = \limsup_{t\to\infty} \frac{1}{t}\ \ln\left(\frac{S^{\delta}_t}{S^{\delta}_0}\right)}$$

Theorem 8 *For* $\boldsymbol{S^{\delta} \in \mathcal{V}^+_x}$

$$\boldsymbol{g^{\delta} \leq g^{\delta_*}.}$$

- "pathwise best" in the long run

  Karatzas & Shreve (1998), Pl. (2004),

  Karatzas & Kardaras (2007)

Pl. (2004)

Definition 9 *Nonnegative portfolio* S^{δ} **outperforms systematically** $S^{\tilde{\delta}}$ *if*

(i) $S_0^{\delta} = S_0^{\tilde{\delta}}$;

(ii) $P\left(S_t^{\delta} \geq S_t^{\tilde{\delta}}\right) = 1$

(iii) $P\left(S_t^{\delta} > S_t^{\tilde{\delta}}\right) > 0.$

- "relative arbitrage" Fernholz & Karatzas (2005)

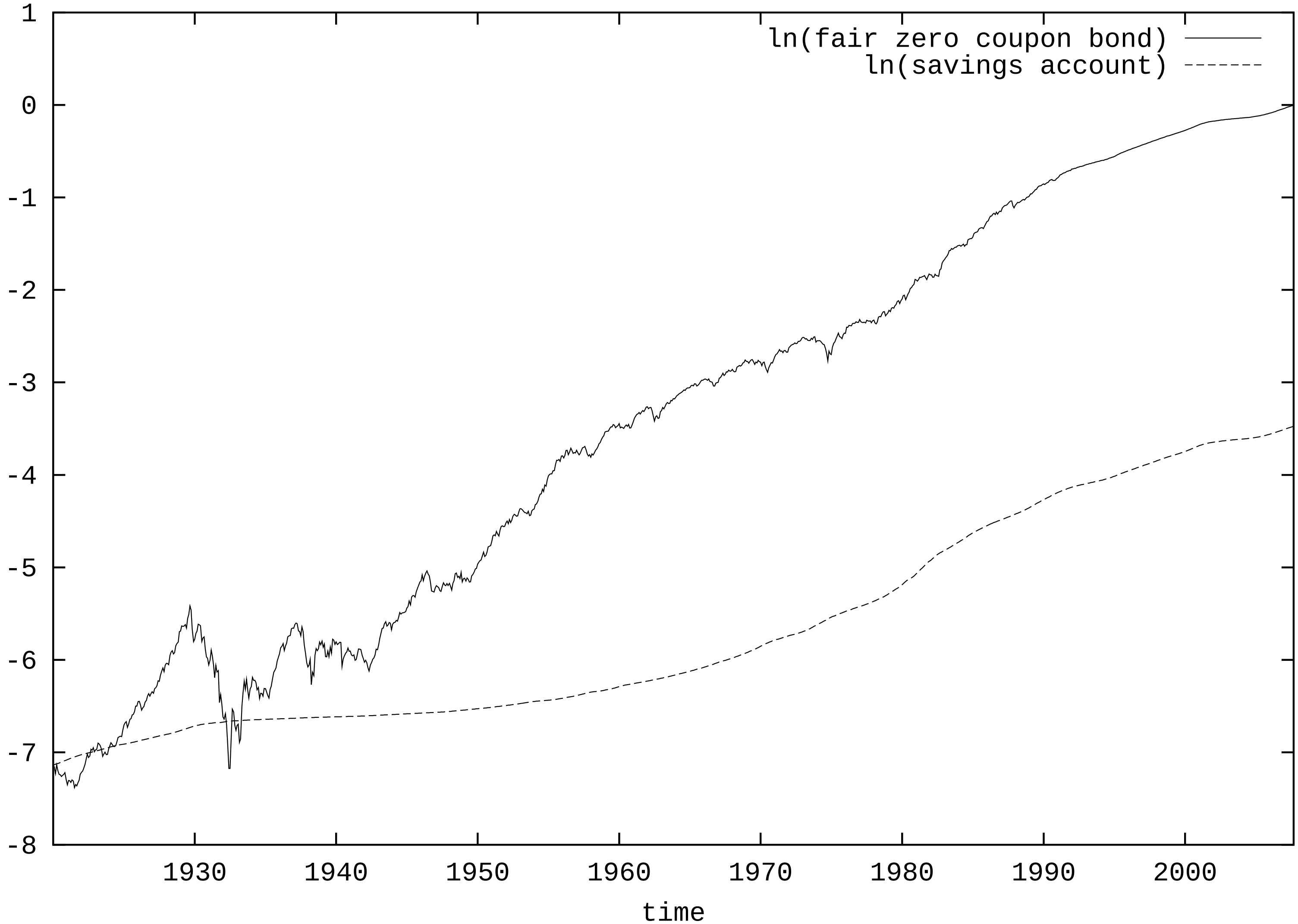

Figure 14: Logarithms of fair zero coupon bond and savings account.

Theorem 10 *Numeraire portfolio cannot be outperformed systematically.*

- **portfolio ratio**

$$A^{\delta}_{t,h} = \frac{S^{\delta}_{t+h}}{S^{\delta}_{t}}$$

- **expected growth**

$$g^{\delta}_{t,h} = E_t\left(\ln\left(A^{\delta}_{t,h}\right)\right)$$

$$t, h \geq 0$$

- **derivative of expected growth**

$S^{\underline{\delta}} \in \mathcal{V}_x^+$

S^{δ} nonnegative

S^{δ_ε} perturbed $S^{\underline{\delta}}$

$$A_{t,h}^{\delta_\varepsilon} = \varepsilon\, A_{t,h}^{\delta} + (1-\varepsilon)\, A_{t,h}^{\underline{\delta}}$$

for $t, h \geq 0,\ \varepsilon > 0$

$$\left.\frac{\partial g_{t,h}^{\delta_\varepsilon}}{\partial \varepsilon}\right|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(g_{t,h}^{\delta_\varepsilon} - g_{t,h}^{\underline{\delta}} \right)$$

Definition 11 $S^{\underline{\delta}}$ growth optimal *if*

$$\frac{\partial g_{t,h}^{\delta_\varepsilon}}{\partial \varepsilon}\bigg|_{\varepsilon=0} \leq 0$$

for all $t, h \geq 0$ *and nonnegative* S^δ.

- alternative definition to **expected log-utility**

  Kelly (1956)
  Hakansson (1971)
  Merton (1973a)
  Roll (1973)
  Markowitz (1976)

Theorem 12 *The numeraire portfolio is growth optimal.*

Strong Arbitrage

- market participants can only exploit arbitrage
- **limited liability**

$\Longrightarrow$ nonnegative total wealth of each market participant

Definition 13 *A nonnegative* S^δ *is a* strong arbitrage *if* $S_0^\delta = 0$ *and*

$$P\left(S_t^\delta > 0\right) > 0.$$

Pl. (2002)-mathematical arguments

Loewenstein & Willard (2000)-economic arguments

Theorem 14 *There is no strong arbitrage.*

$\Longrightarrow$ there is no pricing based on strong arbitrage

- Delbaen & Schachermayer (1998)

  free lunches with vanishing risk (FLVR) may exist

- Loewenstein & Willard (2000)

  free snacks & cheap thrills may exist

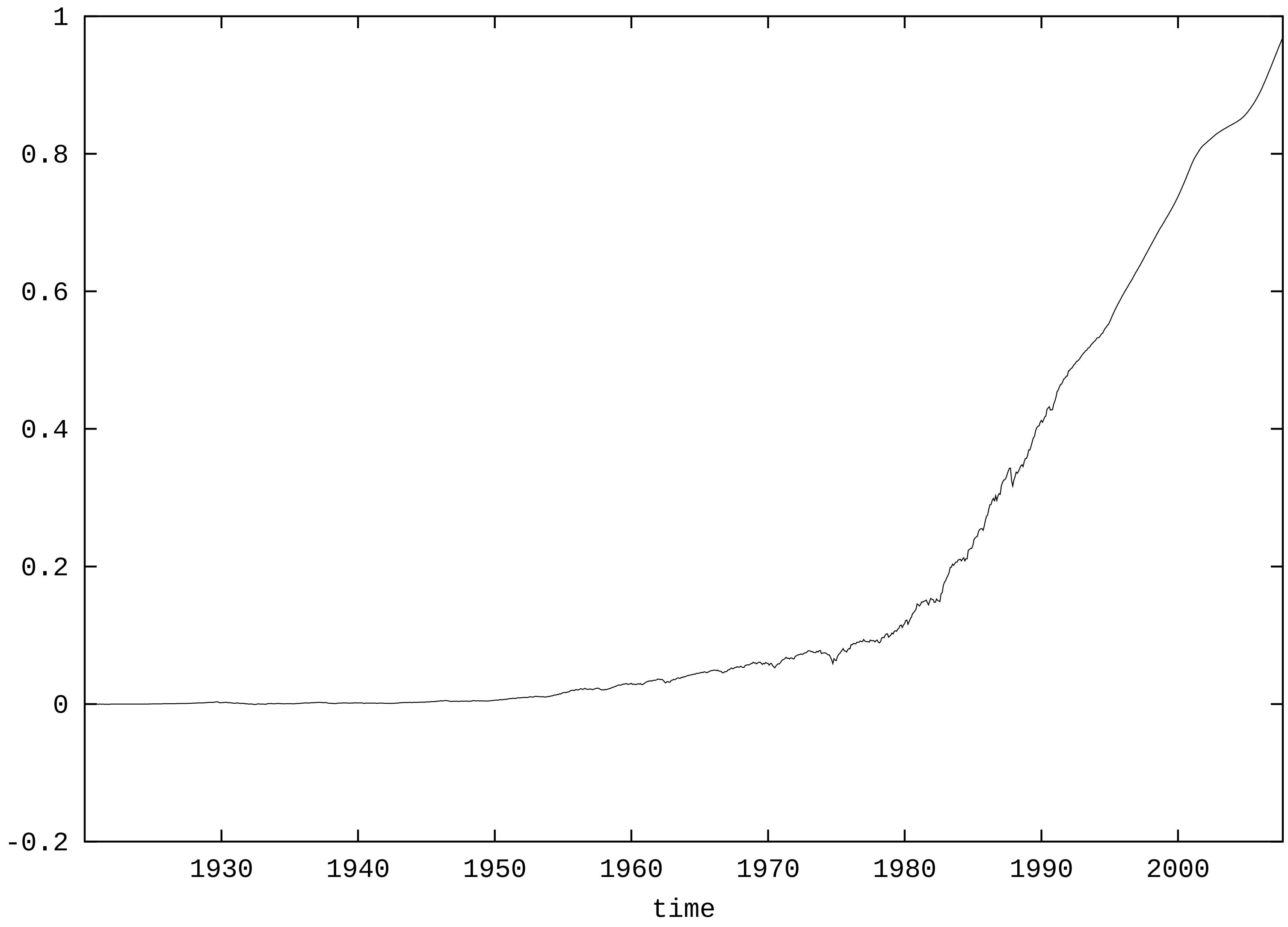

Figure 15: $P(t, T)$ minus savings account.

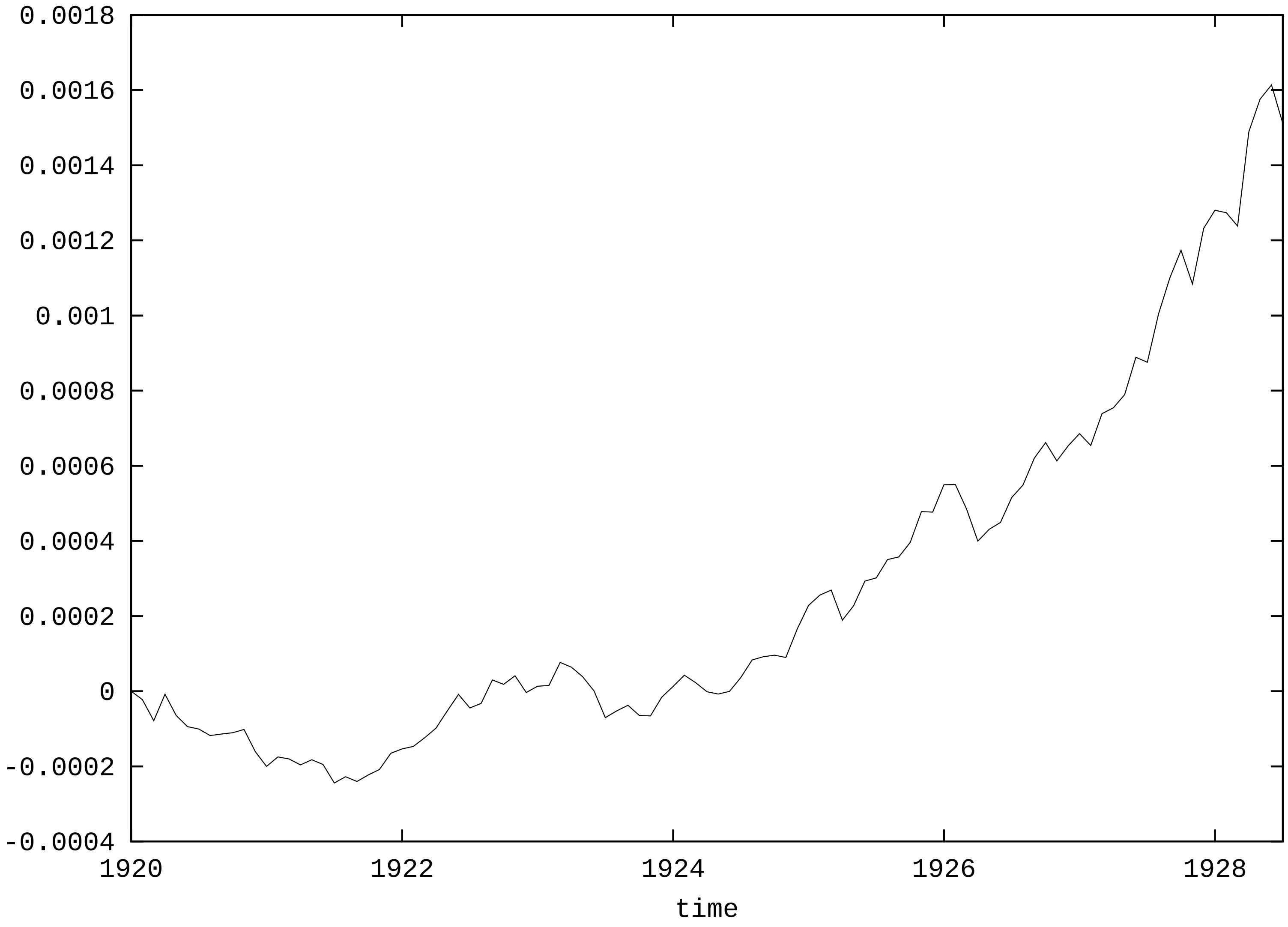

Figure 16: $P(t,T)$ minus savings account.

- Under BA candidate risk neutral Q may ***not*** be equivalent to P since its Radon-Nikodym derivative may be a strict supermartingale.

- Under BA existence of equivalent risk neutral probability measure is more a **mathematical convenience** than an economic necessity.

References

Becherer, D. (2001). The numeraire portfolio for unbounded semimartingales. *Finance Stoch.* **5**, 327–341.

Bühlmann, H. & E. Platen (2003). A discrete time benchmark approach for insurance and finance. *ASTIN Bulletin* **33**(2), 153–172.

Cochrane, J. H. (2001). *Asset Pricing*. Princeton University Press.

Constatinides, G. M. (1992). A theory of the nominal structure of interest rates. *Rev. Financial Studies* **5**, 531–552.

Debreu, G. (1959). *Theory of Value*. Wiley, New York.

Delbaen, F. & W. Schachermayer (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.* **312**, 215–250.

Duffie, D. (2001). *Dynamic Asset Pricing Theory* (3rd ed.). Princeton, University Press.

Fernholz, E. R. & I. Karatzas (2005). Relative arbitrage in volatility-stabilized markets. *Annals of Finance* **1**(2), 149–177.

Goll, T. & J. Kallsen (2003). A complete explicit solution to the log-optimal portfolio problem. *Ann. Appl. Probab.* **13**(2), 774–799.

Hakansson, N. H. (1971). Capital growth and the mean-variance approach to portfolio selection. *J. Financial and Quantitative Analysis* **6**(1), 517–557.

Harrison, J. M. & D. M. Kreps (1979). Martingale and arbitrage in multiperiod securities markets. *J. Economic Theory* **20**, 381–408.

Ingersoll, J. E. (1987). *Theory of Financial Decision Making*. Studies in Financial Economics. Rowman and Littlefield.

Karatzas, I. & C. Kardaras (2007). The numeraire portfolio in semimartingale financial models. *Finance Stoch.* **11**(4), 447–493.

Karatzas, I. & S. E. Shreve (1998). *Methods of Mathematical Finance*, Volume 39 of *Appl. Math.* Springer.

Kelly, J. R. (1956). A new interpretation of information rate. *Bell Syst. Techn. J.* **35**, 917–926.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Rev. Econom. Statist.* **47**, 13–37.

Loewenstein, M. & G. A. Willard (2000). Local martingales, arbitrage, and viability: Free snacks and cheap thrills. *Econometric Theory* **16**(1), 135–161.

Long, J. B. (1990). The numeraire portfolio. *J. Financial Economics* **26**, 29–69.

Markowitz, H. (1976). Investment for the long run: New evidence for an old rule. *J. Finance* **XXXI**(5), 1273–1286.

Merton, R. C. (1973a). An intertemporal capital asset pricing model. *Econometrica* **41**, 867–888.

Merton, R. C. (1973b). Theory of rational option pricing. *Bell J. Econ. Management Sci.* **4**, 141–183.

Platen, E. (2002). Arbitrage in continuous complete markets. *Adv. in Appl. Probab.* **34**(3), 540–558.

Platen, E. (2004). A benchmark framework for risk management. In *Stochastic Processes and Applications to Mathematical Finance*, pp. 305–335. Proceedings of the Ritsumeikan Intern. Symposium: World Scientific.

Platen, E. (2006). A benchmark approach to finance. *Math. Finance* **16**(1), 131–151.

Platen, E. & D. Heath (2006). *A Benchmark Approach to Quantitative Finance*. Springer Finance. Springer.

Roll, R. (1973). Evidence on the ”Growth-Optimum” model. *J. Finance* **28**(3), 551–566.

Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *J. Economic Theory* **13**, 341–360.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *J. Finance* **19**, 425–442.