Replication Based Pricing of Default Contingent Claims

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AGENDA

Brief review of the standard model for single name credit default swaps and similar default contingent claims.

Show how to exactly replicate default contingent claims with (many) standard CDS and relate this to arbitrage based pricing.

Show how to super-replicate default contingent claims with (a few) standard CDS and relate this to arbitrage bounds on pricing.

The standard modeling framework for single name credit default swaps and related default contingent claims (Bloomberg function CDSW $<$GO$>$) is carried out as if the following assumptions are valid:

1. Default free interest rates are deterministic.

2. Recovery in case of default is a deterministic fraction of par.

3. Perfect markets, including no default on CDS contracts.

4. Default happens in a wholly unpredictable manner, with the "risk neutral" probability of default during the short time interval $(t, t+\epsilon)$ given by $\lambda(t)\epsilon$, given no default until time t. Here, $\lambda(t)$ is assumed known as of the present, which is denoted by $t = 0$.

With these assumptions in place, we can choose a functional form for $\lambda(t)$, e.g. piecewise constant, calibrate this function to observable CDS quotes, and value all default contingent cash flows by computing their risk neutral expectation and discounting at the current default free interest rates.

Does this modeling framework really make sense?

It seems to work in practice – there are trillions of dollars of CDS out there, generally being "marked to market" in this framework.

Can we get it to work in theory?

AS IF!

It turns out the standard pricing approach is supported by traditional arbitrage arguments in a much more general setting.

We will retain assumptions 1 and 2, but completely relax the assumptions on the default generating process.

Consider a discrete time economy, with a day being the smallest unit of time, in line with conventions in both CDS and money markets.

Two minor simplifications: no accounting for weekends/holidays, and we assume that CDS contracts have daily premium payments.

We will scale interest rates and CDS premiums to be on a per day basis.

Denote the default time as $\tau \in \mathbb{N}$. A pure default contingent claim is characterized by two deterministic functions of time: c_n and R_n; where our convention is that the claim pays out $\$c_n$ per day for $n \leq \tau$, and $\$R_n$ at $n = \tau$. There will be some "maturity date" N, such that $c_n = R_n = 0, \forall n > N$.

A key example of a pure default contingent claim is a standard CDS of maturity N with notional Q, where $c_n = -S_N Q$ and $R_n = LQ, \forall n \leq N$, where $L \in (0, 1]$ is the assumed fractional loss given default on the underlying name and S_N is the market CDS premium for maturity N.

Default contingent claims that require model based valuation arise naturally from trading of standard CDS, hedging bonds with CDS, and/or trading single name CDS against an index.

Fix an underlying name and a "target" default contingent claim represented by $\{c_n, R_n\}$ with maturity N. Assume that we have CDS available for all maturities $n = 1, 2, ..., N$, with premium quotes $\{S_1, S_2, ..., S_N\}$.

Goal: Find a static CDS replication strategy, given by writing protection on notionals of $\{Q_1, Q_2, ..., Q_N\} \in \mathbb{R}^N$, along with an amount to be deposited in a money market account at time zero, $M_0 \in \mathbb{R}$, such that the target cash flows are replicated regardless of when the underlying name defaults.

Strategy 1: A binomial type of backward recursion, where for each day the risk is that default may or may not happen on that day and the available controls are the quantity of CDS maturing on this day and the amount of money to hold in the money market account, all assuming that default has not yet happened.

If the default time $\tau > n$, the money market account grows by one day of interest and is affected by payment of target coupon and receipt of CDS premium payments (either of which could be negative):

$$M_{n+1} = M_n (1 + r_n) - c_{n+1} + \sum_{k=n+1}^{N} S_k Q_k \tag{1}$$

If $\tau = n$, we need to have just enough money to settle the target claim, after accounting for terminal CDS payments:

$$M_n - L \sum_{k=n}^{N} Q_k - R_n = 0 \tag{2}$$

We can take the difference between expression (2) evaluated at two consecutive days to isolate the value of Q_n:

$$Q_n = \frac{R_n^{'} - M_n^{'}}{L} \tag{3}$$

where the prime denotes a forward first difference in time.

Substituting (3) into (1) and once again taking a first difference and simplifying result in the following linear second order difference equation for the survival contingent money market account balance:

$$M_{n-1}^{''} - r_n M_{n-1}^{'} - \frac{S_n}{L} M_n^{'} - r_{n-1}^{'} M_{n-1} = -\frac{S_n}{L} R_n^{'} - c_n^{'} \tag{4}$$

Using the natural terminal conditions that $M_N = M_{N+1} = 0$, we can recursively solve (4) to obtain the initial required money market account balance, M_0, and along the way we can use (3) at each step to get the static portfolio of CDS that will replicate the target claim.

Since each CDS in the replicating portfolio is costless at inception, the no arbitrage value of the target claim is given by M_0.

We made no assumptions on the nature of the randomness of the default time in this derivation, so M_0 will be compatible with any arbitrage free default dynamics consistent with the initial CDS curve, and in particular it is consistent with the standard market model.

The case of the target claim being a risky annuity is worth special consideration. We have $R_n \equiv 0$, and $c_n = \chi_{\{n \leq N\}}$, and so the right hand side of (4) becomes $\delta_{n,N}$, which suggests that the risky annuity value, as a function of its maturity, is the Green's function of the ODE.

To derive the forward equation for the risky annuity value, let H_n denote the risk neutral probability of survival to time n. By the replication argument above, the risk neutral probability measure will be unique.

Direct valuation of a risky annuity to maturity $n \geq 1$ gives:

$$A_n = \sum_{k=1}^{n} P_k H_{k-1} \tag{5}$$

Taking the first difference in n and re-arranging gives:

$$H_n = \frac{A_{n+1} - A_n}{P_{n+1}} \tag{6}$$

The balance equation for a CDS contract of maturity n can be written as:

$$S_n A_n = L \sum_{k=1}^{n} P_k \left(H_{k-1} - H_k\right) \tag{7}$$

Taking the first difference in n, substituting in for H_n and simplifying gives:

$$A_n'' + r_{n+1}A_{n+1}' + \frac{S_{n+1}}{L}A_n' + \frac{S_n'}{L}A_n = 0 \tag{8}$$

where we note that $A_0 = 0$ and $A_1 = P_1$ (and S_0 can be set to an arbitrary finite amount in the calculation of A_2).

The advantage of the forward equation is that we can solve for the present value of risky annuities of all maturities in a single "sweep". With these in hand, we can represent the value of an arbitrary target claim as:

$$M_0 = \sum_{n=1}^{\infty} A_n \left(-\frac{S_n}{L}R_n' - c_n'\right) \tag{9}$$

Strategy 2: Attempt to set up and solve a system of simultaneous equations characterizing the replicating portfolio.

We can usefully differentiate between N+1 default times, given by $\tau = 1, \tau = 2, ..., \tau = N, \tau > N$. Conditional on any such default time, we can compute the present value of all cash flows relating to the target claim and the replicating CDS portfolio. Equating each resulting present value with the initial value of the money market account gives rise to N+1 linear equations in N+1 unknowns.

While direct solution of the resulting linear system is not computationally efficient for the valuation and hedging of a single contingent claim, it is worth exploring for its general properties, and as a starting point for super-replication in incomplete markets.

For each $\tau = m \leq N$ we have the restriction:

$$M_0 = \sum_{j=1}^{m} P_j \left(\sum_{k=j}^{N} S_k Q_k - c_j \right) - P_m \left(L \sum_{k=m}^{N} Q_k + R_m \right) \tag{10}$$

where the discount factors are found as $P_n = \prod_{i=1}^{n} (1 + r_i)^{-1}$, and for $\tau > N$ we need:

$$M_0 = \sum_{j=1}^{N} P_j \left(\sum_{k=j}^{N} S_k Q_k - c_j \right) \tag{11}$$

We can organize the N+1 equations in N+1 unknowns on the last page into the matrix form $Ax = b$, where

$$\begin{aligned} x &= [Q_1, Q_2, ..., Q_N, M_0]' \\ b_n &= P_n R_n + \sum_{j=1}^{\min(n,N)} P_j c_j \\ A_{m,n} &= \left(\sum_{j=1}^{\min(m,n,N)} P_j \right) S_n - P_m L \chi_{\{m \leq n \leq N\}} \\ A_{m,N+1} &= -1 \end{aligned} \tag{12}$$

As long as there are no arbitrage opportunities in the CDS prices, this system of equations will have a unique solution, of the form

$$x = A^{-1}b \tag{13}$$

Note that the last element of the vector x is the replication based value of the target claim (M_0), whereas the first N elements represent the replicating CDS portfolio.

We can (if we so choose) interpret the elements of the last row of the matrix A^{-1} as (minus) the risk neutral probability of default on each date.

These probabilities are independent of the target claim, so (8) represents a general solution to the replication based pricing problem.

Consider the alternative problem of finding the combination of a static CDS portfolio for a given set of maturities along with a trajectory of the survival contingent money market balance that will minimize the initial cost while always providing at least enough cash to settle the target default contingent claim.

Specifically, we have credit default swaps with maturities $\{m_1, m_2, ..., m_K\}$ and premiums $\{S_1, S_2, ..., S_K\}$, where $1 \leq K \leq N$, and where we use S_k and Q_k as shorthand notation for S_{m_k} and Q_{m_k}, and where we assume for convenience that $m_K = N$.

The no-default dynamics of the money market account balance is now:

$$M_{n+1} = M_n\,(1 + r_n) - c_{n+1} + \sum_{k=1}^{K} S_k Q_k \chi_{\{m_k \geq n+1\}} \tag{14}$$

and in case of default we have the inequality:

$$M_n - L \sum_{k=1}^{K} Q_k \chi_{\{m_k \geq n\}} - R_n \geq 0 \tag{15}$$

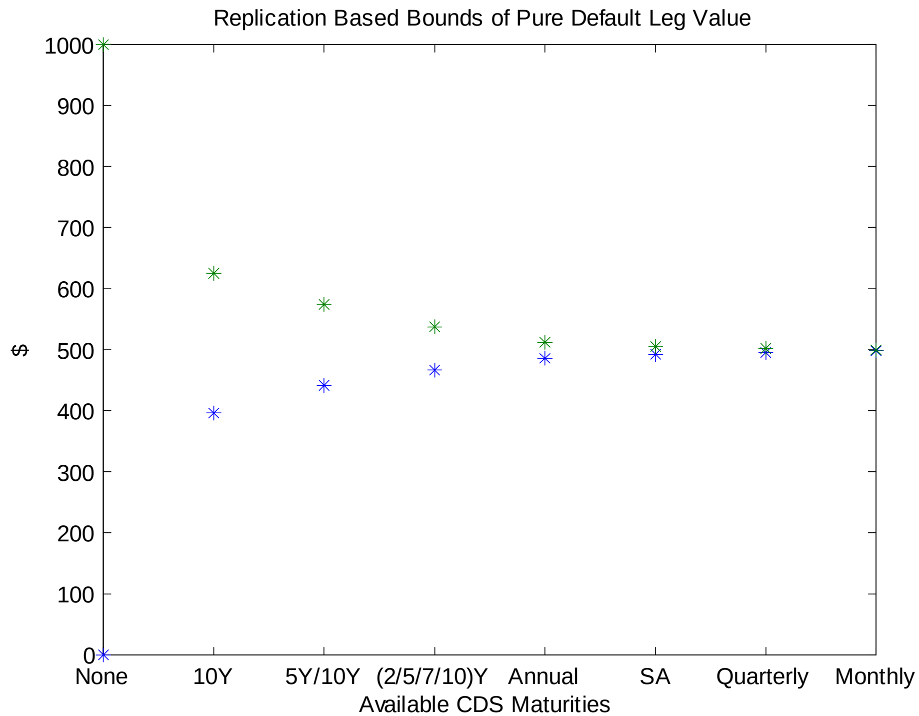
The goal is to minimize M_0, subject to (9) and (10), for an optimal choice of $\{Q_1, Q_2, ..., Q_K\}$.

The general approach to the super-replication problem is to consider a modified version of the matrix A, call it $\bar{A}$, where we only include the $K+1$ columns corresponding to the available hedging instruments, i.e. $\{m_1, m_2, ..., m_K, N+1\}$.

We now have the following optimization problem to solve:

$$\min \quad x_{N+1} \qquad s.t. \quad \bar{A}x \geq b \tag{16}$$

As this is a linear program, it can be solved efficiently with standard LP software.



The graph shows the cost of sub- and super-replication strategies for a 10Y pure default leg, paying $1,000 upon default. The CDS curve is linearly increasing from 1% to 5% and interest rates are flat at 5%.

The difference between the cost of super-replication and exact replication measures how far a target claim lies from the span generated by the pay-offs of the available credit default swaps.

We can turn a pure default leg into a pure premium leg (risky annuity) with a single CDS, so the picture is representative of our ability to replicate any single CDS leg.

The cost of super-replication is sub-additive, so it will always be cheaper to super-replicate a portfolio than its components.

We can obtain good bounds on the value of claims with a reasonable set of available CDS contracts (we don't really need daily maturities!).