Modeling of Dependence Structures in Risk Management and Solvency

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Structure

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2. Copulas
3. Dependent uncorrelated risks and their impact on risk measures
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1. Risk Measurement under Solvency II

Solvency II - Aim

- To harmonize the right of supervision in insurances throughout the EU
- To bring it in line with the supervision rules for credit institutes (Basel II)
- Development of a solvency system which represents the real risks of an insurer

- **S**olvency **C**apital **R**equirement (SCR) is defined in terms of Sandström (2006) as the difference between an appropriate risk measure and equity capital per risk (expected value as base factor plus surcharges (see principle of premium calculation)).

- The SCR is a multiple of the standard deviation of risk when managing normally distributed risks.

- The SCR for the whole risk is calculated by the second root of sum of squares of single SCR's. [(→ square root formula of NAIC = **N**ational **A**ssociation of **I**nsurance **C**ommissioners / IAA = **I**nternational **A**ctuarial **A**ssociation; **German standard model** of ***GDV*** = German Insurance Association ***/ BaFin*** = Federal Financial Supervisory Authority)].

Definition (Coherent Measure of Risk):

A *risk measure* R on the set $\mathfrak{D}(R) \subseteq \mathcal{Z}$ of real-valued random variables (risks) is called *coherent* in terms of ARTZNER, DELBAEN, EBER and HEATH (1999) if it satisfies the following four axioms:

(1) A risk measure R is called *positively homogeneous*, if

$$R(cX) = c\,R(X) \text{ for all } c \geq 0 \text{ and } X \in \mathcal{Z};$$

(2) A risk measure R is called *translation invariant*, if for all $X \in \mathcal{Z}$ and $c \in \mathbb{R}$:

$$R(X + c) = R(X) + c;$$

(3) A risk measure R is called *monotone*, if for any two random variables $X, Y \in \mathcal{Z}$

$$X \leq Y \text{ implies } R(X) \leq R(Y);$$

(4) A risk measure R is called *subadditive*, if

$$R(X + Y) \leq R(X) + R(Y) \text{ for all } X, Y \in \mathcal{Z}.$$

The international discussion about risk measure to calculation of the **solvency capital** (IAA, DAV, SST) with regard to the aggregate claim S concentrates on

***Value at Risk*:** $VaR_\alpha(S) := q_{1-\alpha}(S) = F_S^{-1}(1-\alpha) = \inf\left\{x \in \mathbb{R} \,\middle|\, F_S(x) \geq 1-\alpha\right\};$

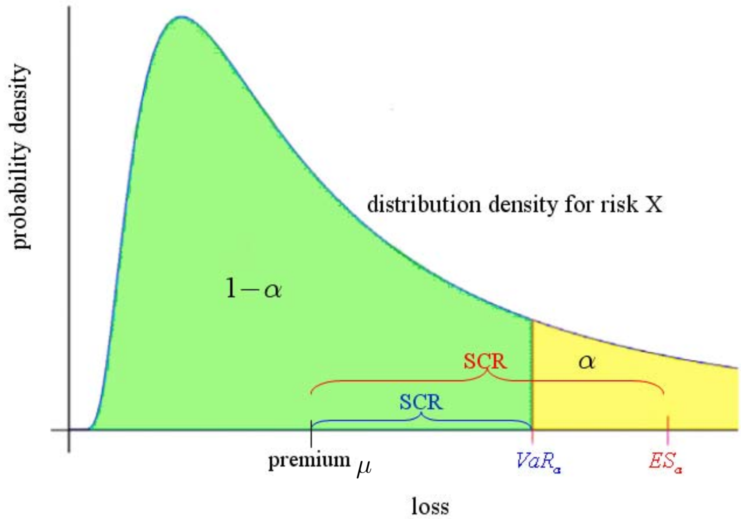
[which is also called

***P**robable **M**aximum **L**oss* (PML) with a return period $1/\alpha$

in the actuarial practice]

and

***Expected Shortfall*:** $ES_\alpha(S) := E\left(S \,\middle|\, S \geq VaR_\alpha(S)\right).$



ES_α : "average" of all values above VaR_α

SCR, SCR : safety loading

The **S**olvency **C**apital **R**equirement (SCR) for every individual risk i, $SCR_{\alpha}(i)$, is defined as the difference between the **V**alue **at R**isk (VaR) and expected value (net premium income),

$$SCR_{\alpha}(i) = VaR_{\alpha}(i) - \mu_i = k_{\alpha} \cdot \sigma_i .$$

Therefore, the *total Solvency Capital Requirement*, SCR_{Gesamt}, (which is called **square root formula**) in the case of normal distributions is given by

$$SCR_{gesamt} = \sqrt{\sum_{i=1}^{n} SCR_i^2 + 2\sum_{i<j}\sum \rho_{ij} \cdot SCR_i \cdot SCR_j}$$

$$= \sqrt{\sum_{i=1}^{n} \left(R(X_i) - \mu_i\right)^2 + 2\sum_{i<j}\sum \rho_{ij} \left(R(X_i) - \mu_i\right)\left(R(X_j) - \mu_j\right)}$$

in consideration of for example of pairwise correlation between risk X_i and risk X_j

$$\rho_{ij} = E(X_i X_j) - E(X_i)E(X_j).$$

The concept of the "square root formula" has the following background: let $VaR_\alpha(i)$ be the **V**alue **at** **R**isk (VaR) to the risk level α for every individual risk X_i, then we can write

$$VaR_\alpha(i) = \mu_i + k_\alpha \sigma_i\,,\; k_\alpha \in \mathbb{R},$$

where μ_i is the expected value ($\mu_i \in \mathbb{R}$) and σ_i denotes the standard deviation ($\sigma_i \geq 0$).

If the risks X_i are normally distributed, then

$$k_\alpha = \Phi^{-1}(1-\alpha) \quad \text{with} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) du = \int_{-\infty}^{x} \varphi(u)\, du\,,$$

where k_α is independent from μ_i and σ_i for $i = 1, \ldots, n$.

Similar: The Expected Shortfall (ES) for normally distributed individual risk X_i is given by

$$ES_\alpha(i) = \mu_i + \frac{e^{-\frac{(k_\alpha)^2}{2}}}{\sqrt{2\pi}\alpha}\sigma_i = \mu_i + \tau_\alpha \sigma_i \text{ with } k_\alpha = \Phi^{-1}(1-\alpha).$$

Comment:

- If the risks are normally distributed then the Value at Risk is a coherent risk measure for $k_\alpha \geq 0 \left(\Leftrightarrow \alpha \leq \frac{1}{2} \right)$.
- If the risks are normally distributed then the mutual dependence structure of the risks is completely and explicitly determined through the pairwise correlations.

Conclusion:

The square root formula is consistent with the definition of Solvency Capital Requirement for both risk measures if the risks are normally distributed.

But:

- If the risks are not normally distributed then the Value at Risk is ***not*** in general a coherent risk measure.
- If the risks are not normally distributed then the premium principle (standard deviation principle) of the square root formula is ***not*** coherent.
- If the risks are not normally distributed then the mutual dependence structure of the risks is ***not*** explicitly determined through the pairwise correlations.

2. Copulas

Definition:
A *copula* is a function C of d variables on the unit d-cube $[0,1]^d$ with the following properties:

1. the range of C is the unit interval $[0,1]$;
2. $C(\mathbf{u})$ is zero for all $\mathbf{u}$ in $[0,1]^d$ for which at least one coordinate equals zero;
3. $C(\mathbf{u}) = u_k$ if all coordinates of $\mathbf{u}$ are 1 except the k-th one;
4. C is d-increasing in the sense that for every $\mathbf{a} \leq \mathbf{b}$ in $[0,1]^d$ the measure $\Delta C_{\mathbf{a}}^{\mathbf{b}}$ assigned by C to the d-box $[\mathbf{a},\mathbf{b}] = [a_1, b_1] \times \cdots \times [a_d, b_d]$ is nonnegative, i.e.

$$\Delta C_{\mathbf{a}}^{\mathbf{b}} := \sum_{(\varepsilon_1,\cdots,\varepsilon_n)\in\{0,1\}^d} (-1)^{\sum_{i=1}^{d}\varepsilon_i} C\left(\varepsilon_1 a_1 + (1-\varepsilon_1) b_1, \cdots, \varepsilon_d a_d + (1-\varepsilon_d) b_d\right) \geq 0.$$

In other words:

A copula C is a multivariate distribution function of a random vector that has continuous uniform margins.

Sklar's Theorem:

Let H denote a n-dimensional distribution function with margins $F_1,\ldots,F_n$. Then there exists a n-copula C such that for all real $(x_1,\ldots,x_n)$,

$$H(x_1,\ldots,x_n) = C\left(F_1(x_1),\ldots,F_n(x_n)\right).$$

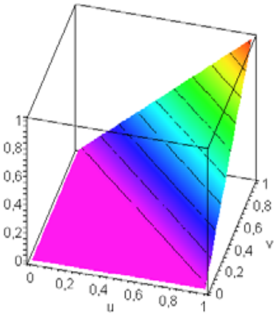
If all the margins are continuous, then the copula is unique, and is determined uniquely on the ranges of the marginal distribution functions otherwise. Moreover, the converse of the above statement is also true. If we denote by $F_1^{-1},\ldots,F_n^{-1}$ the generalized inverses of the marginal distribution functions, then for every $(u_1,\ldots,u_n)$ in the unit n-cube,

$$C(u_1,\ldots,u_n) = H\left(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n)\right).$$

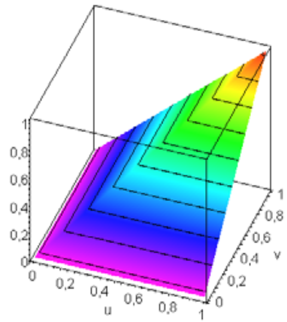
Fréchet-Hoeffding-bounds:

$$\max\left\{\sum_{i=1}^{n} u_i + 1 - n, 0\right\} =: \mathcal{W}^n(\mathbf{u}) \leq \mathbf{C}(u_1,\ldots,u_n) \leq \mathcal{M}^n(\mathbf{u}) := \min\{u_1,\ldots,u_n\},$$

not a copula for $n > 2$



always a copula



Definition (Comonotonicity):

The random variables (risks) $X_1, \ldots, X_n \in \mathcal{Z}$ are called **comonotonic** (*perfectly positive dependent*), if they have the Fréchet-Hoeffding upper bound as copula:

$$\mathcal{M}^n(\mathbf{u}) = \min\{u_1, \ldots, u_n\} \text{ for any } \mathbf{u} \in I^n \text{ and } n \in \mathbb{N}.$$

Definition (Countermonotonicity):
The random variables (risks) $X, Y \in \mathcal{Z}$ are called **countermonotonic** (*perfectly negative dependent*), if they have the Fréchet-Hoeffding lower bound as copula:

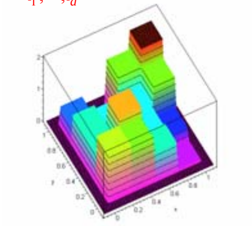
$$\mathcal{W}^2(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\} \text{ for any } u_1, u_2 \in I^2.$$

Wide-spread fallacy: Comonotonic risks increase total risk, countermonotonic risks decrease total risk (→ diversification effect)

$$\boxed{\textbf{This is wrong in general!}}$$

Definition:

Let $d, n \in \mathbb{N}$; define intervals $I_{i_1,\cdots,i_d}(n) := \bigtimes_{j=1}^{d} \left(\frac{i_j - 1}{n}, \frac{i_j}{n} \right]$ for all possible choices $i_1, \ldots, i_d \in N_n := \{1, \ldots, n\}$. For every d-tuple $(i_1, \ldots, i_d) \in N_n{}^d$, let $a_{i_1,\cdots,i_d}(n)$ be a non-negative real number with the property



$$\sum_{(i_1,\cdots,i_d)\in J(i_k)} a_{i_1,\cdots,i_d}(n) = \frac{1}{n}$$

for all $k \in \{1, \ldots, d\}$ and $i_k \in \{1, \ldots, n\}$, with $J(i_k) := \{(j_1, \ldots, j_n) \in N_n{}^d \mid j_k = i_k\}$, then the function $c_n := n^d \sum_{(i_1,\cdots,i_d)\in N_n{}^d} a_{i_1,\cdots,i_d}(n)\, \mathbb{1}_{I_{i_1,\cdots,i_d}(n)}$ is the density of a d-dimensional copula, called *grid-type copula* with parameters $\{a_{i_1,\cdots,i_d}(n) | (i_1, \ldots, i_d) \in N_n{}^d\}$. Here $\mathbb{1}_A$ denotes the indicator random variable of the event *A*, as usual.

It is easy to see that in case of an absolutely continuous *d*-dimensional copula C, with continuous density

$$c(u_1,\ldots,u_d)=\frac{\partial^d}{\partial u_1\ldots\partial u_d}C(u_1,\ldots,u_d),\ (u_1,\ldots,u_d)\in(0,1)^d,$$

c can be approximated arbitrarily close by a density of a grid-type copula. The classical *multivariate mean-value-theorem* of calculus tells us here that we only have to choose

$$a_{i_1,\ldots,i_d}(n):=\int_{\frac{i_d-1}{n}}^{\frac{i_d}{n}}\ldots\int_{\frac{i_1-1}{n}}^{\frac{i_1}{n}}c(u_1,\ldots,u_d)\,du_1\ldots du_d=\Delta C_{\boldsymbol{\alpha}_n}^{\boldsymbol{\beta}_n},\quad i_1,\ldots,i_d\in N_n$$

with $\alpha_{nk}=\dfrac{i_k-1}{n},\ \beta_{nk}=\dfrac{i_k}{n},\ k=1,\ldots,d.$

Lemma:
Let $U_1,\ldots,U_d$ be independent standard uniformly distributed random variables and let f_d and F_d denote the density and cumulative distribution function of $S_d := \sum_{i=1}^{d} U_i$, resp., for $d \in \mathbb{N}$. Then

$$f_d(x) = \frac{1}{2(d-1)!}\sum_{k=0}^{d}(-1)^k \binom{d}{k}(x-k)^{d-1}\operatorname{sgn}(x-k)\ \mathbb{1}_{[0,d]}(x)$$

$$F_d(x) = \frac{1}{2d!}\sum_{k=0}^{d}(-1)^k \binom{d}{k}\Big((-k)^d + (x-k)^d \operatorname{sgn}(x-k)\Big)\ \mathbb{1}_{[0,d]}(x) + \mathbb{1}_{(d,\infty]}(x)$$

for $x \in \mathbb{R}$.

This follows e.g. from USPENSKY (1937), Example 3, p. 277, who attributes this result already to Laplace.

Theorem:
Let $(X_1, \cdots, X_d)$ be a random vector whose joint cumulative distribution function is given by a grid-type copula with density $c_n := \sum_{(i_1,\ldots,i_d)\in N_n{}^d} a_{i_1,\ldots,i_d}(n) \mathbb{1}_{I_{i_1,\ldots,i_d}(n)}$. Then the density and cumulative distribution function $\tilde{f}_d(n;\bullet)$ and $\tilde{F}_d(n;\bullet)$, resp., for the sum $S_d := \sum_{i=1}^{d} X_i$ is given by

$$\left.\begin{aligned} \tilde{f}_d(n;x) &= n \sum_{(i_1,\cdots,i_d)\in N_n{}^d} a_{i_1,\cdots,i_d}(n) \cdot f_d\left(nx + d - \sum_{j=1}^{d} i_j\right) \\ \tilde{F}_d(n;x) &= \sum_{(i_1,\cdots,i_d)\in N_n{}^d} a_{i_1,\cdots,i_d}(n) \cdot F_d\left(nx + d - \sum_{j=1}^{d} i_j\right) \end{aligned}\right. \quad \text{for } x \in \mathbb{R},$$

with density f_d and cumulative distribution functions F_d from the Lemma above.

3. Dependent *uncorrelated* risks and their impact to risk measures

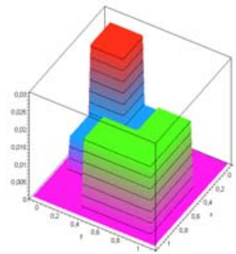
Example:

Consider a grid-type copula with 9 subsquares, i.e. $d=2$ and $n=3$. Let the weights $a_{ij}(n)$ for the copula density be given in matrix form as

$$A(3)=\left[a_{ij}(3)\right]=\begin{bmatrix} a & b & 1/3-a-b \\ c & 1-4a-2b-2c & -2/3+4a+2b+c \\ 1/3-a-c & -2/3+4a+b+2c & 2/3-3a-b-c \end{bmatrix},$$

with suitable real numbers $a,b,c\in[0,1/3]$. It follows that the covariance of the corresponding random variables X_1, X_2 is given by

$$E(X_1X_2)-E(X_1)E(X_2)=\frac{1}{9}\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij}(3)(i-2)(j-2)=0,$$



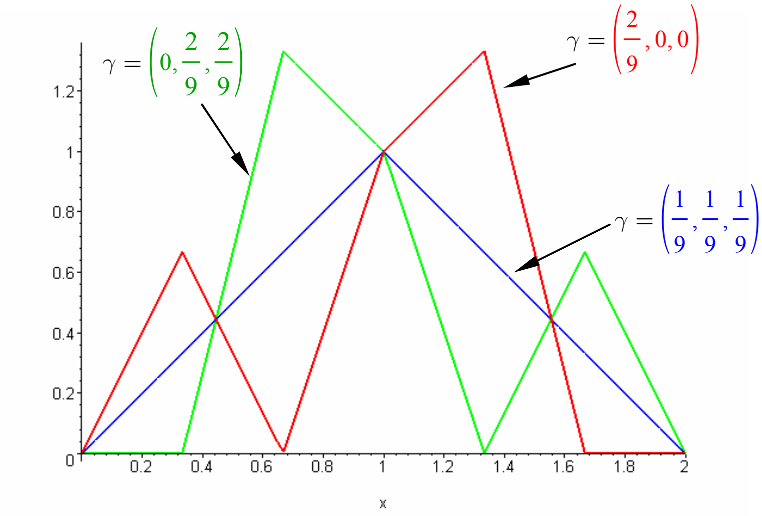
i.e. the risks X_1 and X_2 are ***uncorrelated but dependent***.

The density and cumulative distribution function of the aggregated risk $S_2 := X_1 + X_2$ are thus, by the above Theorem, given by

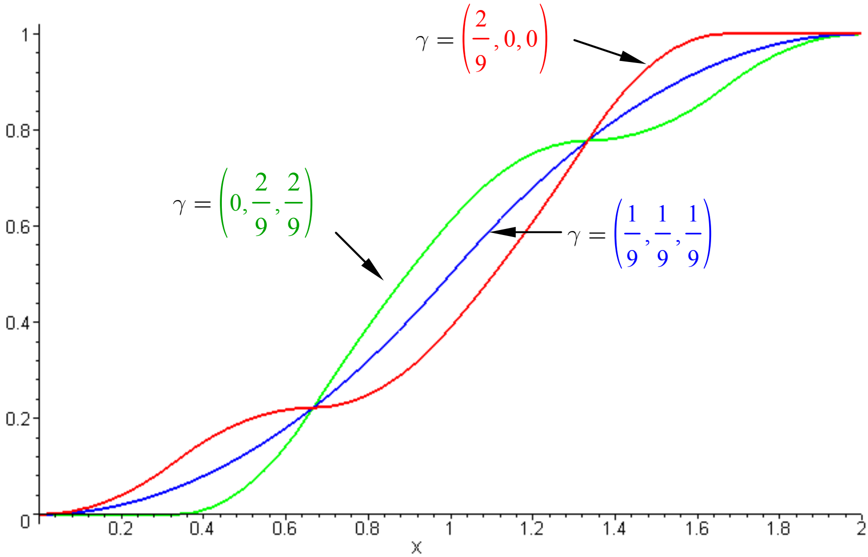
$$\tilde{f}_2(3;\gamma;x) = \begin{cases} 9ax, & 0 \le x \le \frac{1}{3} \\ 3(2a - \{b+c\}) + 9(-a + \{b+c\})x, & \frac{1}{3} \le x \le \frac{2}{3} \\ 9(4a + 3\{b+c\}) - 10 + 3(5 - 18a - 12\{b+c\})x, & \frac{2}{3} \le x \le 1 \\ 32 - 9(16a + 7\{b+c\}) + 9(14a + 6\{b+c\} - 3)x, & 1 \le x \le \frac{4}{3} \\ -28 + 3(52a + 19\{b+c\}) + 3(6 - 33a - 12\{b+c\})x, & \frac{4}{3} \le x \le \frac{5}{3} \\ 6(2 - 9a - 3\{b+c\}) + 3(-2 + 9a + 3\{b+c\})x, & \frac{5}{3} \le x \le 2 \\ 0, & \text{otherwise;} \end{cases}$$

$$\tilde{F}_2(3;\gamma;x)=\begin{cases}0, & x\le 0\\ \dfrac{9a}{2}x^2, & 0\le x\le \dfrac{1}{3}\\ \dfrac{9}{2}(-a+\{b+c\})x^2+3(2a-\{b+c\})x+\dfrac{1}{2}(-2a+\{b+c\}), & \dfrac{1}{3}\le x\le \dfrac{2}{3}\\ \dfrac{5}{2}(3-18a-12\{b+c\})x^2+(-10+36a+27\{b+c\})x+ \\ \qquad +\dfrac{1}{6}(20-66a-57\{b+c\}), & \dfrac{2}{3}\le x\le 1\\ \dfrac{9}{2}(-3+14a+6\{b+c\})x^2+(32-144a-63\{b+c\})x+ \\ \qquad +\dfrac{1}{6}(-106+237a+213\{b+c\}), & 1\le x\le \dfrac{4}{3}\\ \dfrac{9}{2}(2-22a-4\{b+c\})x^2+(-28+156a-57\{b+c\})x+ \\ \qquad +\dfrac{1}{6}(134-726a-267\{b+c\}), & \dfrac{4}{3}\le x\le \dfrac{5}{3}\\ \dfrac{3}{2}(-6+9a+3\{b+c\})x^2+3(4-6a-18\{b+c\})x+ \\ \qquad +(-11+54a+18\{b+c\}), & \dfrac{5}{3}\le x\le 2\\ 1, & x\ge 2.\end{cases}$$

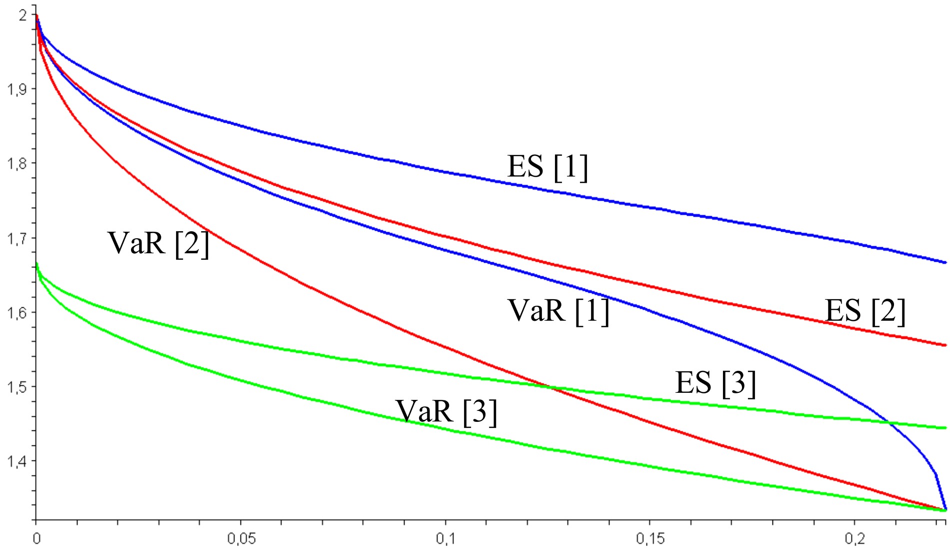
The following graphs show three different densities $\tilde{f}_2(3;\gamma;\bullet)$ and cumulative distribution functions $\tilde{F}_2(3;\gamma;\bullet)$ for the sum $S_2 = X_1 + X_2$, for various choices of $\gamma = (a,b,c)$.



Plots of $\tilde{f}_2(3;\gamma;\bullet)$ for various choices of γ



Plots of $\tilde{F}_2(3;\gamma;\bullet)$ for various choices of γ



Value at Risk and Expected Shortfall for positive dependence [1], independence [2], and negative dependence [3]

For the example we get for the three cases following values for the risk measures

	positive dependent	independent	negative dependent	square root formula
$VaR_{0.1}$	**1.6838**	**1.5528**	**1.4430**	**1.5657**
$ES_{0.1}$	**1.7892**	**1.7019**	**1.5269**	**1.6364**
$VaR_{0.01}$	**1.9000**	**1.8586**	**1.5960**	**1.6930**
$ES_{0.01}$	**1.9333**	**1.9057**	**1.6225**	**1.7000**
$VaR_{0.005}$	**1.9293**	**1.9000**	**1.6167**	**1.7000**
$ES_{0.005}$	**1.9411**	**1.9167**	**1.6260**	**1.7036**

Value at Risk (VaR) and Expected Shortfall (ES)
with $\alpha = 0.1,\ \alpha = 0.01,$ and $\alpha = 0.005$

4. Dependent risks and their impact to risk measures

Example (risk distribution with *heavy tail*)**:**

Suppose that the risks X and Y follow a Pareto distribution with density

$$f(x) = \lambda(1+x)^{-(1+\lambda)}, \; x \geq 0 \; (\lambda > 0)$$

each.

Then the density g and cumulative distribution function G of the aggregated risk $S_2 := X + Y$ can be explicitly calculated:

$\underline{\lambda = 1/2:}$

Case 1: X and Y are *comonotonic*:

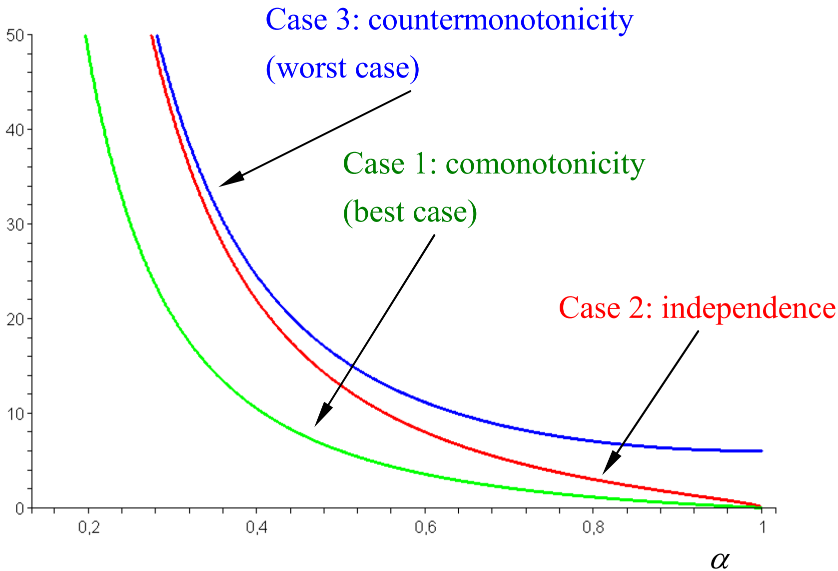
$$g_{S_2}(z) = \frac{1}{4\left(\sqrt{1+z/2}\right)^3} \approx \frac{1}{\sqrt{2}\left(\sqrt{1+z}\right)^3}, \quad z \geq 0;$$

Case 2: X and Y are *independent*:

$$g_{S_2}(z) = \frac{z}{(2+z)^2\sqrt{1+z}} \approx \frac{1}{\left(\sqrt{1+z}\right)^3}, \quad z \geq 0;$$

Case 3: X and Y are *countermonotonic*:

$$g_{S_2}(z) = \frac{4+z-2\sqrt{3+z}}{\sqrt{z+6-4\sqrt{3+z}}\,\sqrt{3+z}\cdot\left(\sqrt{2+z}\right)^3} \approx \frac{1}{\left(\sqrt{1+z}\right)^3}, \quad z \geq 6.$$



Value at Risk with $\lambda = 1/2$ for the three cases

Calculation of the Value at Risk for $S_2 := X + Y$ and $0 < \alpha < 1$:

Case 1: X and Y are *comonotonic*:

$$VaR_\alpha(S_2) = \frac{2}{\alpha^2} - 2;$$

Case 2: X and Y are *independent*:

$$VaR_\alpha(S_2) = \frac{4}{\alpha^2} - 2 - \frac{2}{1+\sqrt{1-\alpha^2}} \sim \frac{4}{\alpha^2} - 4 \;\; (\alpha \to 0);$$

Case 3: X and Y are *countermonotonic*:

$$VaR_\alpha(S_2) = \frac{4}{\alpha^2} - 2 + \frac{4}{(2-\alpha)^2} \sim \frac{4}{\alpha^2} + 2 \;\; (\alpha \to 0).$$

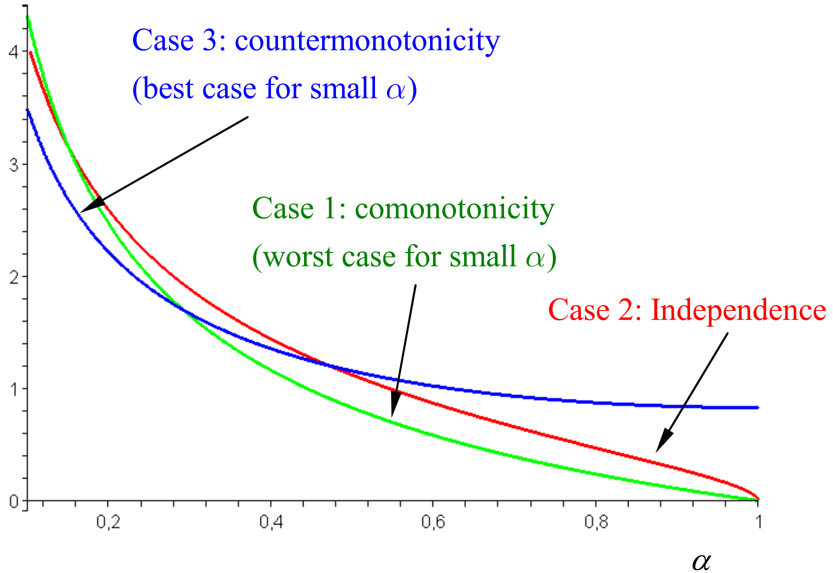
Consequence:

1. The risk measure for **a** portfolio consisting of **two** *independent* or *comonotonic* risks from the same type is *greater than* the sum of the risk measures for **two** portfolios with one risk for every portfolio!

→ No diversification effect!

2. The risk measure for **a** portfolio of **two** *independent* risks of the same type is *asymptotic equivalent* (with large return period) to the risk measure for **a** portfolio of **two** *countermonotonic* risks from the same type!

→ Independence is near to the „worst case“!



Value at Risk with $\lambda = 2$ for the three cases

Calculation of the Value at Risk for $S_2 := X + Y$ and $0 < \alpha < 1$:

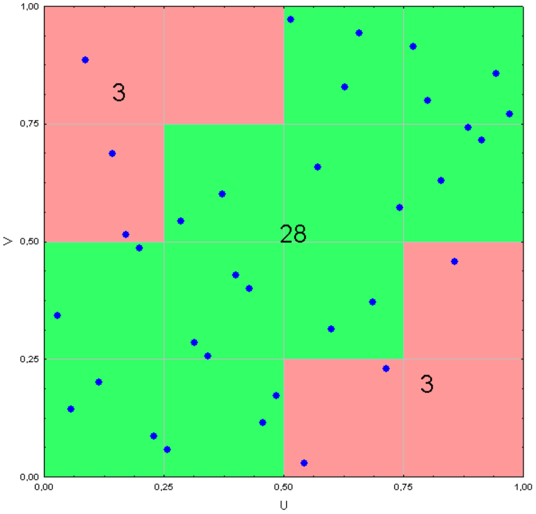
Case 1: X and Y are *comonotonic*:

$$VaR_\alpha(S_2) = \frac{2}{\sqrt{\alpha}} - 2;$$

Case 3: X and Y are *countermonotonic*:

$$VaR_\alpha(S_2) = \frac{2}{\sqrt{\alpha}} \sqrt{\frac{1 + \sqrt{1 - (1-\alpha)^2}}{2 - \alpha}} - 2 \sim \frac{\sqrt{2}}{\sqrt{\alpha}} - 2 \ (\alpha \to 0).$$

6. Implications for DFA and Solvency II



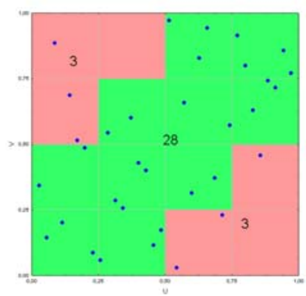
$$\tilde{\mathbf{A}} = \frac{1}{34}\begin{bmatrix} 3 & 3 & 2 & 0 \\ 2 & 4 & 2 & 1 \\ 2 & 2 & 2 & 3 \\ 1 & 0 & 3 & 4 \end{bmatrix}$$

$$= \frac{1}{136}\begin{bmatrix} 12 & 12 & 8 & 0 \\ 8 & 16 & 8 & 4 \\ 8 & 8 & 8 & 12 \\ 4 & 0 & 12 & 16 \end{bmatrix}$$

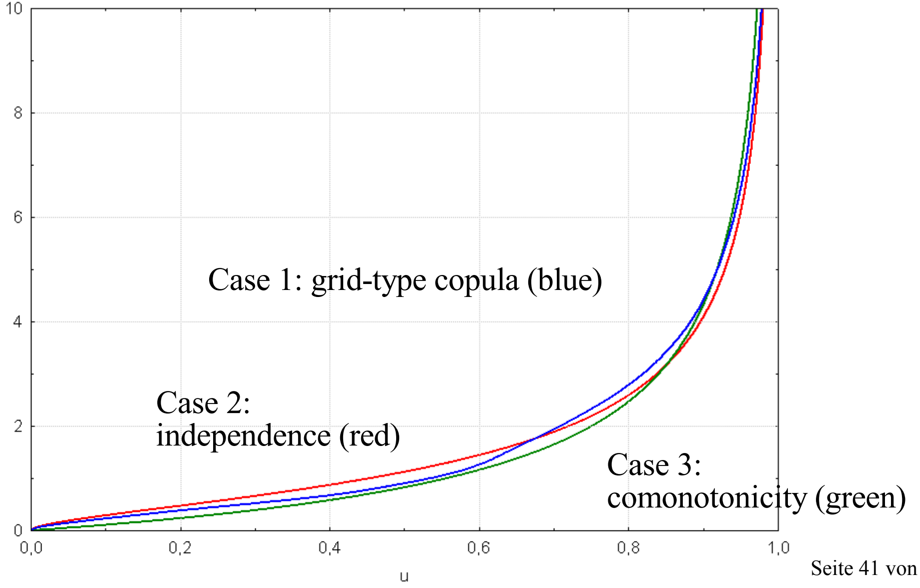
Scatterplot pertaining to an empirical copula for windstorm vs. flooding, from real data

The data can be well fitted to a 4×4 grid-type copula represented by the following weight matrix:

$$\mathbf{A} = \begin{bmatrix} \frac{13}{136} & \frac{12}{136} & \frac{8}{136} & \frac{1}{136} \\ \frac{8}{136} & \frac{15}{136} & \frac{7}{136} & \frac{4}{136} \\ \frac{8}{136} & \frac{7}{136} & \frac{7}{136} & \frac{12}{136} \\ \frac{5}{136} & 0 & \frac{12}{136} & \frac{17}{136} \end{bmatrix}$$



The following graph shows the empirical quantile function for the aggregate risk from a Monte Carlo study with 100 000 simulations using this copula. We assume for simplicity and purposes of comparison that the marginal distributions of windstorm and flooding are of the same Pareto type as above, with $\lambda = 2$.



Thank you for your attention!

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